>>> Topological invariant of 4-manifolds based on a 3-group

>>> *9th Tux Workshop on Quantum Gravity*

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>>> A sketch of the talk

- 3-group and 3-gauge theory
  - ↪ based on *R. Picken and J. Faria Martins*, Diff. Geom. Appl. 29, 179 (2011), arXiv:0907.2566.
- $3BF$ action with constraints
  - ↪ Models with relevant dynamics *T. Radenković and M. Vojinović*, J. High Energy Phys.10, 222 (2019), arXiv:1904.07566.
- Quantization of the topological $3BF$ theory
  - ↪ The construction of the state sum Z and a proof that the $3BF$ state sum is invariant under Pachner moves.
    T. Radenković and M. Vojinović, arXiv: 2201.02572.
  - ↪ the state sum Z is an example of Porter's TQFT for $d = 4$ and $n = 3$
    T. Porter, J. Lond. Math. Soc. (2)58, No. 3, 723 (1998), MR 1678163.
- Pachner move invariance - sketch of the proof
  - ↪ This is a generalization of the state sum based on the classical $2BF$ action with the underlying 2-group structure
    F. Girelli, H. Pfeiffer and E. M. Popescu, Jour. Math. Phys. 49, 032503 (2008), arXiv:0708.3051.
- Conclusions

2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G, \triangleright, \{\_,\_\}_{\mathrm{p}})$

* Groups G, H, and L;
* maps ∂ and δ $(\partial\delta = 1_G)$;
* an action $\triangleright$ of the group G on all three groups;
* a map $\{\_\,,\_\}_{\mathrm{p}}$ called the *Peiffer lifting*:

$$\{\_\,,\_\}_{\mathrm{p}} : H \times H \to L\,.$$

Certain axioms hold true among all these maps:

1. $\delta(\{h_1,h_2\}_{\mathrm{p}}) = \langle h_1\,,h_2\rangle_{\mathrm{p}}\,, \qquad \forall h_1,h_2 \in H\,,$
2. $[l_1,l_2] = \{\delta(l_1)\,,\delta(l_2)\}_{\mathrm{p}}\,, \qquad \forall l_1\,,l_2 \in L$. Here, the notation $[l,k] = lkl^{-1}k^{-1}$ is used;
3. $\{h_1h_2,h_3\}_{\mathrm{p}} = \{h_1,h_2h_3h_2^{-1}\}_{\mathrm{p}}\partial(h_1) \triangleright \{h_2,h_3\}_{\mathrm{p}}\,, \qquad \forall h_1,h_2,h_3 \in H\,;$
4. $\{h_1,h_2h_3\}_{\mathrm{p}} = \{h_1,h_2\}_{\mathrm{p}}\{h_1,h_3\}_{\mathrm{p}}\{\langle h_1,h_3\rangle_{\mathrm{p}}^{-1},\partial(h_1) \triangleright h_2\}_{\mathrm{p}}\,, \qquad \forall h_1,h_2,h_3 \in H\,;$
5. $\{\delta(l),h\}_{\mathrm{p}}\{h,\delta(l)\}_{\mathrm{p}} = l(\partial(h) \triangleright l^{-1})\,, \qquad \forall h \in H\,, \quad \forall l \in L\,.$

>>> The $3BF$ theory

One can now generalize the notion of parallel transport from curves to surfaces and volumes.

* Given a 2-crossed module, one can define a *3-connection*, an ordered triple (α,β,γ), where α, β, and γ are algebra-valued differential forms,

$$\begin{aligned} \alpha &= \alpha^{\alpha}{}_{\mu}\,\tau_{\alpha}\,\mathrm{d}x^{\mu}\,, & \alpha &\in \mathcal{A}^1(\mathcal{M}_4,\mathfrak{g})\,,\\ \beta &= \beta^{a}{}_{\mu\nu}\,t_a\,\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\,, & \beta &\in \mathcal{A}^2(\mathcal{M}_4,\mathfrak{h})\,,\\ \gamma &= \gamma^{A}{}_{\mu\nu\rho}\,T_A\,\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\rho}\,, & \gamma &\in \mathcal{A}^3(\mathcal{M}_4,\mathfrak{l})\,. \end{aligned} \tag{1}$$

* Then introduce the *line, surface and volume holonomies*,

$$g = \mathcal{P}\exp\int_{\gamma}\alpha\,,\quad h = \mathcal{S}\exp\int_{S}\beta\,,\quad l = \mathcal{V}\exp\int_{V}\gamma\,. \tag{2}$$

* The corresponding *fake 3-curvature* $(\mathcal{F},\mathcal{G},\mathcal{H})$ is defined as:

$$\begin{gathered} \mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,,\qquad \mathcal{G} = \mathrm{d}\beta + \alpha\wedge^{\triangleright}\beta - \delta\gamma\,,\\ \mathcal{H} = \mathrm{d}\gamma + \alpha\wedge^{\triangleright}\gamma + \{\beta\wedge\beta\}_{\mathrm{pf}}\,. \end{gathered} \tag{3}$$

>>> The $3BF$ theory

At this point one can construct the so-called $3BF$ theory.

* For a manifold $\mathcal{M}_4$ and the 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright, \{\_\,,\_\}_{\mathrm{pf}})$, that gives rise to 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, one defines the $3BF$ action as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,. \tag{4}$$

    * $3BF$ theory is a topological gauge theory,
    * it is based on the 3-group structure,
    * it is a generalization of an ordinary BF theory for a given Lie group G.
* The physical interpretation of the Lagrange multipliers C and D:
    * the $\mathfrak{h}$-valued 1-form C can be interpreted as the tetrad field if if $H = \mathbb{R}^4$ is the spacetime translation group,

$$C \to e = e^a{}_\mu(x) t_a \mathrm{d}x^\mu \,, \tag{5}$$

    * the $\mathfrak{l}$-valued 0-form D can be interpreted as the set of real-valued matter fields, given some Lie group L:

$$D \to \phi = \phi^A(x) T_A \,. \tag{6}$$

>>> Constrained $3BF$ action

* For a manifold $\mathcal{M}_4$ and the 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\rm pf})$, that gives rise to 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, one defines the $3BF$ action as

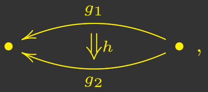
$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,. \tag{7}$$

* Physically relevant models - The constrained $2BF$ actions describing the *Yang-Mills field* and *Einstein-Cartan gravity*, and constrained $3BF$ actions describing the *Klein-Gordon*, *Dirac*, *Weyl* and *Majorana fields* coupled to Yang-Mills fields and gravity in the standard way are formulated.
  ↪ *T. Radenković and M. Vojinović*, J. High Energy Phys.10, 222 (2019), arXiv:1904.07566.

>>> 3-gauge theory

* Curves are labeled with the elements of G, and the elements are composed as

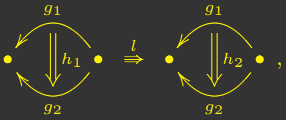
* Surfaces are labeled with the elements $h \in H$. We split the boundary into two curves, the source curve $g_1 \in G$ and the target curve $g_2 \in G$,



so that the surface $h \in H$ satisfies:

$$\partial(h) = g_2 g_1^{-1} \,.$$

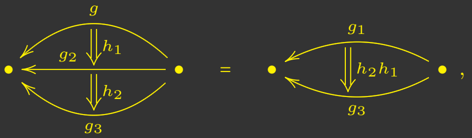
* Volumes are labeled with the elements $l \in L$. We split the boundary into the source surface $\partial_3^-(l) = h_1$ and the target surface $\partial_3^+(l) = h_2$, and the common boundary of h_1 and h_2 we split into the source curve $\partial_2^-(l) = g_1$ and the target curve $\partial_2^+(l) = g_2$,



$$\delta(l) = h_2 h_1^{-1} \,.$$

>>> 3-gauge theory

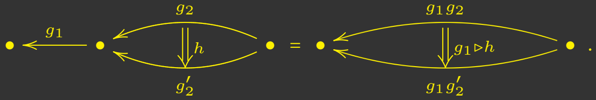
* *Vertical composition of 2-morphisms.* One can compose 2-morphisms (g_1, h_1) and (g_2, h_2) vertically, when they are compatible, when $\partial_2^+(h_1) = \partial_2^-(h_2)$,



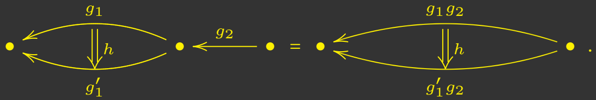
results in a 2-morphism $(g_1, h_2 h_1)$,

$$(g_2, h_2)\#_2(g_1, h_1) = (g_1, h_2 h_1)\,. \tag{8}$$

* *Whiskering.* One can whisker a 2-morphism h with a morphism g_1 by attaching the whisker g_1 to the surface h from the left, such that $\partial_1^-(g_1) = \partial_1^+(h)$,

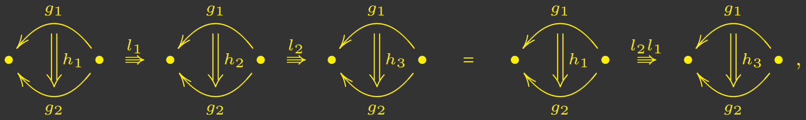


One can whisker g_2 to a surface h from the right, such that $\partial_1^-(h) = \partial_1^+(g_2)$,



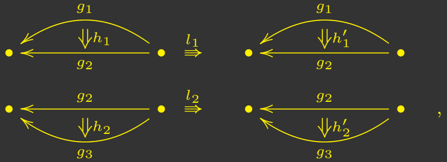
>>> 3-gauge theory

* *Upward composition*. The upward composition of 3-morphisms (g_1, h_1, l_1) and (g_1, h_2, l_2), when they are compatible, when $\partial_3^+(l_1) = \partial_3^-(l_2)$,

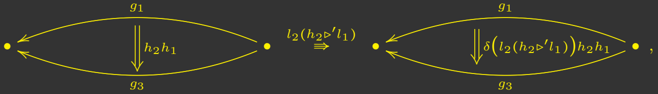


$$(g_1, h_2, l_2)\#_3(g_1, h_1, l_1) = (g_1, h_1, l_2 l_1)\,. \tag{9}$$

* *Vertical composition*. The vertical composition of two 3-morphisms (g_1, h_1, l_1) and (g_2, h_2, l_2), when they are compatible, when $\partial_2^+(l_1) = \partial_2^-(l_2)$,



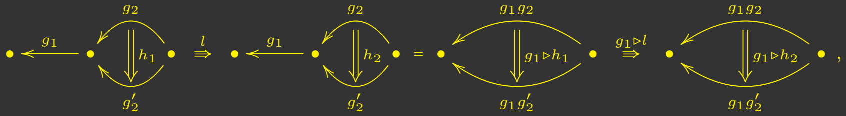
results in a 3-morphism $(g_1, h_2 h_1, l_2(h_2 \triangleright' l_1))$,



$$(g_2, h_2, l_2)\#_2(g_1, h_1, l_1) = (g_1, h_2 h_1, l_2(h_2 \triangleright' l_1))\,. \tag{10}$$

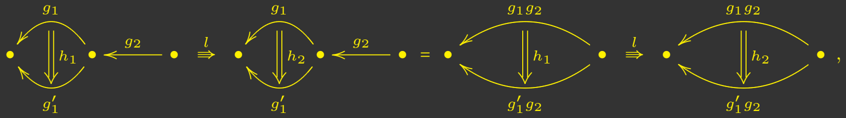
>>> 3-gauge theory

* *Whiskering of the 3-morphisms with morphisms*. Whiskering of a 3-morphism by a morphism from the left is the composition of a volume $l \in L$ and curve $g_1 \in G$ from the left, when they are compatible, when $\partial_1^+(l) = \partial_1^-(g_1)$,



$$g_1 \#_1 (g_2, h_1, l) = (g_1 g_2, g_1 \triangleright h, g_1 \triangleright l) \,. \tag{11}$$

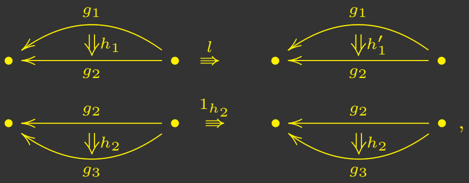
One can whisker a 3-morphism by a morphism from the right, when they are compatible, $\partial_1^-(l) = \partial_1^+(g_2)$,



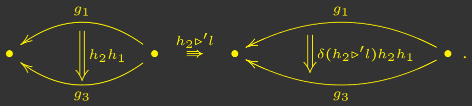
$$(g_1, h_1, l) \#_1 g_2 = (g_1 g_2, h_1, l) \,. \tag{12}$$

>>> 3-gauge theory

* *Whiskering of 3-morphisms with 2-morphisms.* Whiskering of a 3-morphism with a 2-morphisms from below, when they are compatible, $\partial_2^+(l) = \partial_2^-(h_2)$, is formed as a vertical composition of 3-morphisms (g_1, h_1, l) and $(g_2, h_2, 1_{h_2})$,



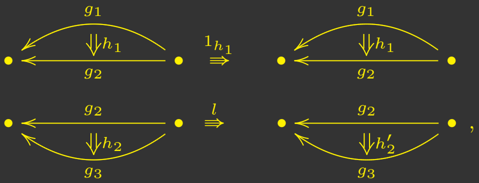
which results in a 3-morphism



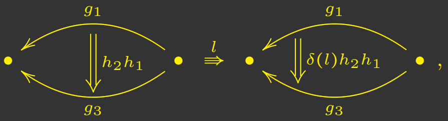
$$(g_1, h_1, l)\#_2(g_2, h_2) = (g_1, h_2 h_1, h_2 \triangleright' l)\,. \tag{13}$$

>>> 3-gauge theory

* Whiskering a 3-morphism by 2-morphism from above, when they are compatible, when $\partial_2^-(l) = \partial_2^+(h_1)$,



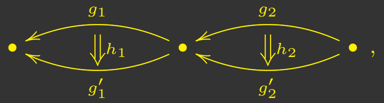
results in a 3-morphism,



$$(g_1, h_1)\#_2(g_2, h_2, l) = (g_1, h_2h_1, l)\,. \tag{14}$$

>>> 3-gauge theory

* *The interchanging 3-arrow*. The horizontal composition of two 2-morphisms h_1 and h_2, when they are compatible, when $\partial_1^-(h_1) = \partial_1^+(h_2)$,

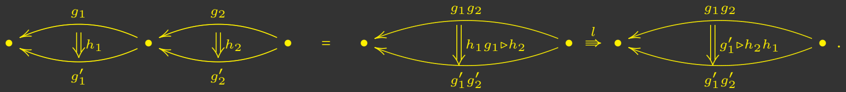


that results in a 3-morphism l, with source surface

$$\partial_3^-(l) = \big((g_1, h_1)\#_1 g_2'\big)\#_2\big(g_1\#_1(g_2, h_2)\big)\,,$$

and target surface

$$\partial_3^+(l) = \big(g_1'\#_1(g_2, h_2)\big)\#_2\big((g_1, h_1)\#_1 g_2\big)\,,$$



One obtains,

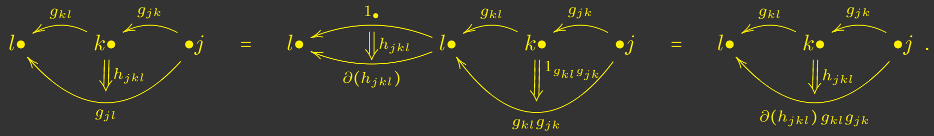
$$(g_1, h_1)\#_1(g_2, h_2) = (g_1 g_2, h_1 g_1 \triangleright h_2, l)\,, \tag{15}$$

where the 3-morphism l is Peiffer lifting $\{h_1, g_1 \triangleright h_2\}_{\mathrm{p}}^{-1}$.

>>> 3-gauge theory

Lemma (1)

Let us consider a triangle, $(jk\ell)$. The edges $(jk), j < k$, are labeled by group elements $g_{jk} \in G$ and the triangle $(jk\ell), j < k < \ell$, by element $h_{jk\ell} \in H$.



(16)

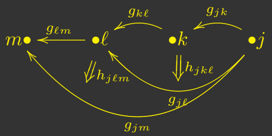
The curve $\gamma_1 = g_{k\ell}g_{jk}$ is the source and the curve $\gamma_2 = g_{j\ell}$ is the target of the surface morphism $\Sigma : \gamma_1 \to \gamma_2$, labeled by the group element $h_{jk\ell}$,

$$g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk} \,. \tag{17}$$

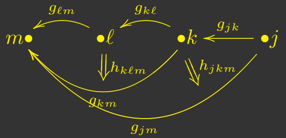
>>> 3-gauge theory

Lemma (2)

Let us consider a tetrahedron, $(jk\ell m)$.



$$= (g_{\ell m}g_{j\ell}, h_{j\ell m})\#_2\big(g_{\ell m}\#_1(g_{k\ell}g_{jk}, h_{jk\ell})\big) = \big(g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell m}(g_{\ell m} \triangleright h_{jk\ell})\big)\,. \tag{18}$$



$$= (g_{km}g_{jk}, h_{jkm})\#_2\big((g_{\ell m}g_{k\ell}, h_{k\ell m})\#_1 g_{jk}\big) = (g_{\ell m}g_{k\ell}g_{jk}, h_{jkm}h_{k\ell m})\,. \tag{19}$$

Moving from surface shown on the diagram (18) to the surface shown on the diagram (19) is determined by the group element $l_{jk\ell m}$,

$$h_{jkm}h_{k\ell m} = \delta(l_{jk\ell m})h_{j\ell m}(g_{\ell m} \triangleright h_{jk\ell})\,. \tag{20}$$

Lemma (3)

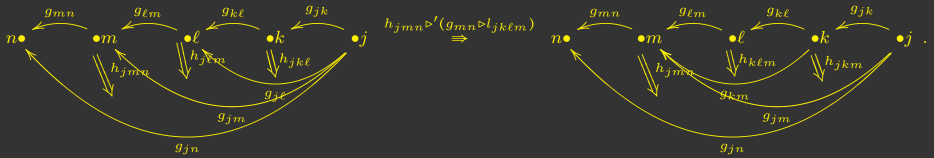
We consider a 4-simplex, $(jk\ell mn)$. We cut the 4-simplex volume along the surface $h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$.

Step 1.

* We move the surface from $h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}$ to $h_{jkm}h_{k\ell m}$ with the 3-arrow $l_{jk\ell m}$.
* To compose the resulting 3-morphism with surface h_{jmn} one must first whisker it from the left with g_{mn}.
* The obtained 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), g_{mn} \triangleright l_{jk\ell m})$ can be whiskered from below with the 2-morphism $(g_{mn}g_{jm}, h_{jmn})$.
* The resulting 3-morphism is

$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}))} \quad (21)$$

$\Sigma_1 \rightarrow \Sigma_2$, $\Sigma_1 = h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$ and $\Sigma_2 = h_{jmn}g_{mn} \triangleright (h_{jkm}h_{k\ell m})$.



Lemma (3)

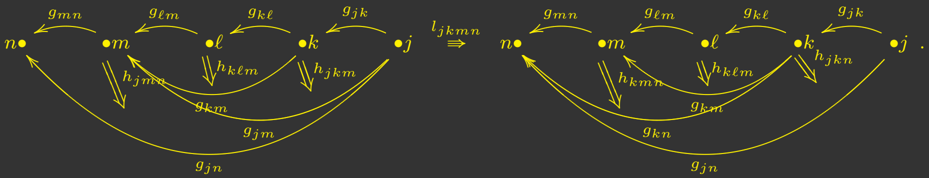
Step 2.

Let us move the surface to $h_{jkn} h_{kmn} g_{m\ell} \triangleright h_{k\ell m}$.

* We consider the 3-morphism $(g_{mn} g_{km} g_{jk}, h_{jmn} g_{mn} \triangleright h_{jkm}, l_{jkmn})$ with the source surface $h_{jmn} g_{mn} \triangleright h_{jkm}$ and target surface $h_{jkn} h_{kmn}$.
* This 3-morphism can be whiskered from above with the 2-morphism $(g_{mn} g_{\ell l} g_{k\ell} g_{jk}, g_{mn} \triangleright h_{k\ell m})$.
* The obtained 3-morphism is

$$\boxed{(g_{mn} g_{\ell m} g_{k\ell} g_{jk}, h_{jmn} g_{mn} \triangleright (h_{jkm} h_{k\ell m}), l_{jkmn})}$$

$\Sigma_1 \to \Sigma_2$, $\Sigma_1 = h_{jmn} g_{mn} \triangleright (h_{jkm} h_{k\ell m})$ and $\Sigma_2 = h_{jkn} h_{kmn} g_{mn} \triangleright h_{k\ell m}$.



(22)

Lemma (3)

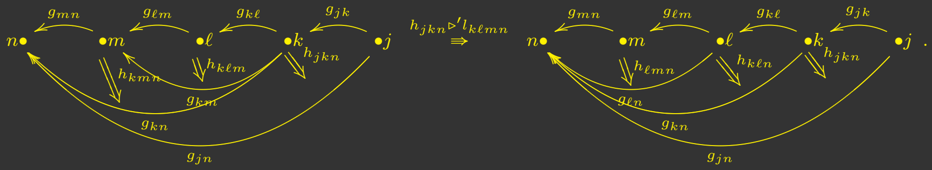
Step 3.

Next, we want to move the surface $h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}$ to surface $h_{jkn}h_{k\ell n}h_{\ell mn}$.

* We whisker the 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}, h_{kmn}g_{mn} \triangleright h_{k\ell m}, l_{k\ell mn})$, with the source surface $h_{kmn}g_{mn} \triangleright h_{k\ell m}$ and target surface $h_{k\ell n}h_{\ell mn}$, with the morphism g_{jk} from the right.
* The obtained the 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{kmn}g_{mn} \triangleright h_{k\ell m}, l_{k\ell mn})$ we whisker with the 2-morphism $(g_{kn}g_{jk}, h_{jkn})$ from below.
* We obtain the 3-morphism

$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}, h_{jkn} \triangleright' l_{k\ell mn})}$$

$\Sigma_1 \to \Sigma_2$, $\Sigma_1 = h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}$ and $\Sigma_2 = h_{jkn}h_{k\ell n}h_{\ell mn}$.



(23)

Lemma (3)

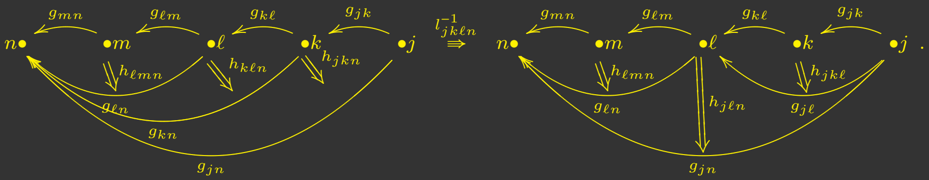
Step 4.

We map the surface $h_{jkn}h_{k\ell n}h_{\ell mn}$ to the surface $h_{j\ell n}g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}$.

* The 3-morphism with the appropriate source and target is constructed by whiskering the 3-morphism $(g_{\ell n}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}, l^{-1}_{jk\ell n})$ with 2-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{\ell mn})$ from above.
* The obtained 3-morphism is

$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}h_{\ell mn}, l^{-1}_{jk\ell n})}$$

$$\Sigma_1 \to \Sigma_2\,, \quad \Sigma_1 = h_{jkn}h_{k\ell n}h_{\ell mn} \text{ and } \Sigma_2 = h_{j\ell n}g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}\,.$$



(24)

Lemma (3)

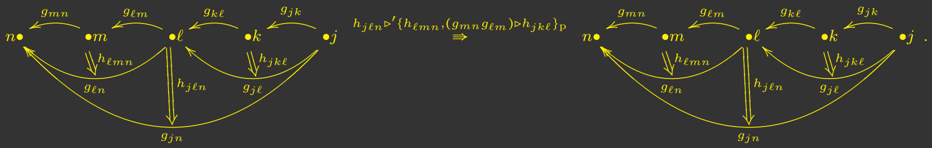
Step 5.

Next we map the surface $h_{j\ell n} g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}$ to the surface $h_{j\ell n} h_{\ell mn} (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}$.

* We use the inverse interchanging 2-arrow composition to map the surface $g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}$ to the surface $h_{\ell mn} (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}$, resulting in the 3-morphism $(g_{mn} g_{\ell m} g_{k\ell} g_{jk}, g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}, \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}})$.
* Next, we whisker the obtained 3-morphism with the 2-morphism $(g_{\ell n} g_{j\ell}, h_{j\ell n})$ from below.
* The obtained 3-morphism with the appropriate source and target surfaces is

$$\boxed{(g_{mn} g_{\ell m} g_{k\ell} g_{jk}, h_{j\ell n} g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}})}$$

$\Sigma_1 \to \Sigma_2$, $\Sigma_1 = h_{j\ell n} g_{\ell n} \triangleright h_{jk\ell} h_{\ell mn}$ and $\Sigma_2 = h_{j\ell n} h_{\ell mn} (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}$.



(25)

>>> 3-gauge theory

Lemma (3)

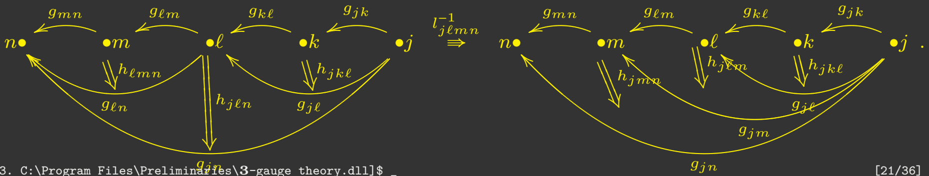
Step 6.

Finally, we construct the 3-morphism that maps the surface $h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m})\triangleright h_{jk\ell}$ to the starting surface $h_{jmn}g_{mn}\triangleright(h_{j\ell m}g_{\ell m}\triangleright h_{jk\ell})$.

* To obtain the 3-morphism with the appropriate source and target surfaces we first move the surface $h_{j\ell n}h_{\ell mn}$ to the surface $h_{jmn}g_{mn}\triangleright h_{j\ell m}$ with the 3-arrow $(g_{mn}g_{\ell m}g_{j\ell}, h_{j\ell n}h_{\ell mn}, l^{-1}_{j\ell mn})$.
* Next, we whisker the 3-morphism $(g_{mn}g_{\ell m}g_{j\ell}, h_{j\ell n}h_{\ell mn}, l^{-1}_{j\ell mn})$ with the 2-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, (g_{mn}g_{\ell m})\triangleright h_{jk\ell})$ from above.
* The obtained 3-morphism

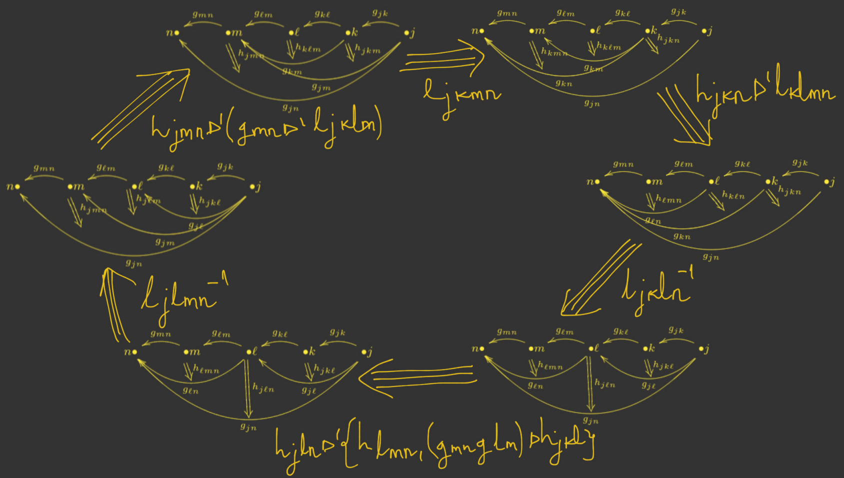
$$\boxed{(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m})\triangleright h_{jk\ell}, l^{-1}_{j\ell mn})}$$

$\Sigma_1\to\Sigma_2$, $\Sigma_1 = h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m})\triangleright h_{jk\ell}$ and $\Sigma_2 = h_{jmn}g_{mn}\triangleright(h_{j\ell m}g_{\ell m}\triangleright h_{jk\ell})$.



>>> 3-gauge theory

Lemma (3)



Lemma (3)

After the upward composition of the 3-morphisms given by the diagrams (21)-(26), the obtained 3-morphism is:

$$
\begin{aligned}
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}, l^{-1}_{j\ell mn})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}h_{\ell mn}, l^{-1}_{jk\ell n})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{kmn}g_{m\ell} \triangleright h_{k\ell m}, h_{jkn} \triangleright' l_{jkmn})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{jkm}h_{k\ell m}), l_{jkmn})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})) \\
&= (g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), l^{-1}_{j\ell mn}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}} \\
&\quad l^{-1}_{jk\ell n}(h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}))\,.
\end{aligned} \tag{27}
$$

The obtained 3-morphism is the identity morphism with source and target surface $\mathcal{V}_1 = \mathcal{V}_2 = h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$,

$$
l^{-1}_{j\ell mn}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}}\, l^{-1}_{jk\ell n}(h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}) = e\,. \tag{28}
$$

>>> Quantization of the topological $3BF$ theory

We want to construct a *state sum model* from the classical S_{3BF} action by the usual *spinfoam quantization procedure*.

$$Z = \int \mathcal{D}\alpha\,\mathcal{D}\beta\,\mathcal{D}\gamma\,\mathcal{D}B\,\mathcal{D}C\,\mathcal{D}D\,\exp\left(i\int_{M_4}\langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}} + \langle D\wedge\mathcal{H}\rangle_{\mathfrak{l}}\right). \tag{29}$$

↪ The formal integration over the Lagrange multipliers B, C, and D leads to:

$$Z = \mathcal{N}\int \mathcal{D}\alpha\,\mathcal{D}\beta\,\mathcal{D}\gamma\;\; \delta(\mathcal{F})\delta(\mathcal{G})\delta(\mathcal{H}). \tag{30}$$

↪ Discretization of the 3-connection:

- $\alpha\in\mathcal{A}^1(\mathcal{M}_4,\mathfrak{g}) \mapsto g_\epsilon\in G$ coloring the edges $\epsilon=(jk)\in\Lambda_1$,
- $\beta\in\mathcal{A}^2(\mathcal{M}_4,\mathfrak{h}) \mapsto h_\Delta\in H$ coloring the triangles $\Delta=(jk\ell)\in\Lambda_2$,
- $\gamma\in\mathcal{A}^3(\mathcal{M}_4,\mathfrak{l}) \mapsto l_\tau\in L$ coloring the tetrahedrons $\tau=(jk\ell m)\in\Lambda_3$.

$$\left.\begin{array}{lcl} \int\mathcal{D}\alpha & \mapsto & \prod_{(jk)\in\Lambda_1}\int_G dg_{jk} \\ \int\mathcal{D}\beta & \mapsto & \prod_{(jk\ell)\in\Lambda_2}\int_H dh_{jk\ell} \\ \int\mathcal{D}\gamma & \mapsto & \prod_{(jk\ell m)\in\Lambda_3}\int_L dl_{jk\ell m} \end{array}\right\} \longrightarrow \text{The disretization of path integral measures.}$$

>>> Quantization of the toplogical $3BF$ theory

↪ The condition $\delta(\mathcal{F})$ is disretized as

$$\delta(\mathcal{F}) = \prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell}), \qquad \delta_G(g_{jk\ell}) = \delta_G\Big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\Big). \tag{31}$$

↪ The condition $\delta(\mathcal{G})$ on the fake curvature 3-form reads

$$\delta(\mathcal{G}) = \prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m}), \tag{32}$$

$$\delta_H(h_{jk\ell m}) = \delta_H\Big(\delta(l_{jk\ell m}) h_{j\ell m}\,(g_{\ell m} \triangleright h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\Big). \tag{33}$$

↪ The condition $\delta(\mathcal{H})$ is disretized as

$$\delta(\mathcal{H}) = \prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn}), \tag{34}$$

$$\delta_L(l_{jk\ell mn}) = \delta_L\Big(l_{j\ell mn}^{-1}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}}\, l_{jk\ell n}^{-1} (h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})\Big). \tag{35}$$

...all off this $\Longrightarrow$

$$Z = \mathcal{N} \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \Bigg(\prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\Bigg)\Bigg(\prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\Bigg)\Bigg(\prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\Bigg). \tag{36}$$

This expression can be made independent of the triangulation if one appropriately chooses the constant factor $\mathcal{N}$.

Definition

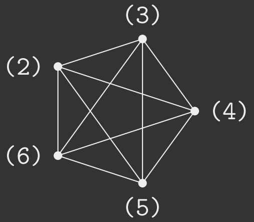
Let $\mathcal{M}_4$ be a compact and oriented combinatorial 4-manifold, and $(L \overset{\delta}{\to} H \overset{\partial}{\to} G, \triangleright, \{\_,\_\}_{\mathrm{pf}})$ be a 2-crossed module. The state sum of *topological higher gauge theory* is defined by

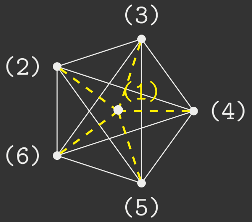
$$\boxed{\begin{aligned} Z \;=\;& |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\,|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|} \\ & \times\left(\prod_{(jk)\in\Lambda_1}\int_G dg_{jk}\right)\left(\prod_{(jk\ell)\in\Lambda_2}\int_H dh_{jk\ell}\right)\left(\prod_{(jk\ell m)\in\Lambda_3}\int_L dl_{jk\ell m}\right) \\ & \times\left(\prod_{(jk\ell)\in\Lambda_2}\delta_G\big(\partial(h_{jk\ell})\,g_{k\ell}\,g_{jk}\,g_{j\ell}^{-1}\big)\right)\left(\prod_{(jk\ell m)\in\Lambda_3}\delta_H\big(\delta(l_{jk\ell m})h_{j\ell m}\,(g_{\ell m}\triangleright h_{jk\ell})\,h_{k\ell m}^{-1}\,h_{jkm}^{-1}\big)\right) \\ & \times\left(\prod_{(jk\ell mn)\in\Lambda_4}\delta_L\big(l_{j\ell mn}^{-1}\,h_{j\ell n}\triangleright'\{h_{\ell mn},(g_{mn}g_{\ell m})\triangleright h_{jk\ell}\}_{\mathrm{p}}\,l_{jk\ell n}^{-1}(h_{jkn}\triangleright' l_{k\ell mn})l_{jkmn}h_{jmn}\triangleright'(g_{mn}\triangleright l_{jk\ell m})\big)\right). \end{aligned}} \tag{37}$$

Here $|\Lambda_0|$ denotes the number of vertices, $|\Lambda_1|$ edges, $|\Lambda_2|$ triangles, $|\Lambda_3|$ tetrahedrons, and $|\Lambda_4|$ 4-simplices of the triangulation.

↪ *T. Radenković and M. Vojinović,* arXiv: 2201.02572.

>>> 1 ↔ 5 Pachner move





$1 \leftrightarrow 5$

	l.h.s.	r.h.s
M_0		(1)
M_1		(12), (13), (14), (15), (16)
M_2		(123), (124), (125), (126), (134), (135), (136), (145), (146), (156)
M_3		(1234), (1235), (1236), (1245), (1246), (1256), (1345), (1346), (1356), (1456)
M_4	(23456)	(13456), (12456), (12356), (12346), (12345)

>>> $1 \leftrightarrow 5$ Pachner move

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.h.s.	5	10	10	5	1
r.h.s.	6	15	20	15	5

Right side

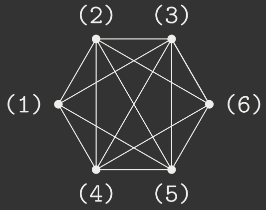
$$Z^{1\leftrightarrow 5}_{\text{right}} = |G|^{-11}|H|^{-4}|L|^{-1}\int_{G^5}\prod_{(jk)\in M_1} dg_{jk}\int_{H^{10}}\prod_{(jk\ell)\in M_2} dh_{jk\ell}\int_{L^{10}}\prod_{(jklm)\in M_3} dl_{jklm} \cdot\left(\prod_{(jk\ell)\in M_2}\delta_G(g_{jk\ell})\right)\left(\prod_{(jk\ell m)\in M_3}\delta_H(h_{jk\ell m})\right)\left(\prod_{(jk\ell mn)\in M_4}\delta_L(l_{jk\ell mn})\right)Z_{\text{remainder}}\,, \tag{38}$$

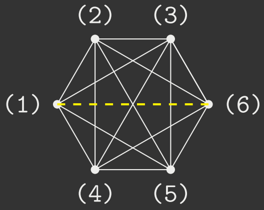
Left side

$$Z^{1\leftrightarrow 5}_{\text{left}} = |G|^{-5}|H|^{0}|L|^{-1}\delta_L(l_{23456})Z_{\text{remainder}}\,. \tag{39}$$

The $Z_{\text{remainder}}$ denotes the part of the state sum that is the same on both sides of the move, and thus irrelevant for the proof of invariance.

>>> $2 \leftrightarrow 4$ Pachner move





$2 \leftrightarrow 4$

	l.h.s.	r.h.s
M_0		
M_1		(16)
M_2		$(126), (136), (146), (156)$
M_3	(2345)	$(1236), (1246), (1256), (1346), (1356), (1456)$
M_4	$(23456), (12345)$	$(12346), (12356), (12456), (13456)$

>>> $2 \leftrightarrow 4$ Pachner move

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.h.s.	6	14	16	9	2
r.h.s.	6	15	20	14	4

Right side

$$Z^{2\leftrightarrow 4}_{left} = |G|^{-8}|H|^{-1}|L|^{-1} \int_L dl_{2345}\delta_H(h_{2345})\left(\prod_{(jk\ell mn)\in M_4}\delta_L(l_{jk\ell mn})\right) Z_{\mathtt{remainder}}\,, \tag{40}$$

Left side

$$Z^{2\leftrightarrow 4}_{right} = |G|^{-11}|H|^{-3}|L|^{-1} \int_G dg_{16} \int_{H^4} dh_{126}dh_{136}dh_{146}dh_{156} \int_L dl_{1236}dl_{1246}dl_{1256}dl_{1346}dl_{1356}dl_{1456} \left(\prod_{(jk\ell)\in M_2}\delta_G(g_{jk\ell})\right)\left(\prod_{(jk\ell m)\in M_3}\delta_H(h_{jk\ell m})\right)\left(\prod_{(jk\ell mn)\in M_4}\delta_L(l_{jk\ell mn})\right) Z_{\mathtt{remainder}}\,. \tag{41}$$

>>> Proof of $2 \leftrightarrow 4$ Pachner move invariance

* On the <u>*left hand side of the move*</u> one has the following integrals and the integrand,

$$\int_L dl_{2345} \delta_H(h_{2345}) \delta_L(l_{23456}) \delta_L(l_{12345}). \tag{42}$$

We integrate out l_{2345} using $\delta_L(l_{12345})$. The δ-function $\delta_H(h_{2345})$ now reads,

$$\delta_H(h_{2345}) = \delta_H(e). \tag{43}$$

The remaining δ-function $\delta_L(l_{23456})$, reads

$$\begin{aligned} \delta_L(l_{23456}) = \; & \delta_L\Big({l_{2456}}^{-1} {l_{2346}}^{-1} l_{2356} (h_{256} g_{56} \triangleright {h_{125}}^{-1}) \triangleright' g_{56} \triangleright l_{1235} \big(h_{256} g_{56} \triangleright {h_{125}}^{-1} g_{56} \triangleright h_{135}\big) \triangleright' \\ & \Big((g_{35} \triangleright h_{123} h_{356}^{-1}) \triangleright' l_{3456}) \{g_{56} \triangleright h_{345}, (g_{56} g_{45} g_{34}) \triangleright h_{123}\}_{\mathrm{p}}^{-1} (g_{56} \triangleright h_{345} (g_{56} g_{45}) \triangleright (h_{123} h_{234}^{-1}) h_{456}^{-1}) \triangleright' \\ & \{h_{456}, (g_{56} g_{45}) \triangleright h_{234}\}_{\mathrm{p}} \Big) (h_{256} g_{56} \triangleright {h_{125}}^{-1}) \triangleright' g_{56} \triangleright l_{1345} \\ & (h_{256} g_{56} \triangleright {h_{125}}^{-1} g_{56} \triangleright h_{145}) \triangleright' ((g_{56} g_{45}) \triangleright l_{1234})^{-1} (h_{256} g_{56} \triangleright {h_{125}}^{-1}) \triangleright' g_{56} \triangleright l_{1245}^{-1} \Big). \end{aligned} \tag{44}$$

Finally, the l.h.s. reads:

$$\boxed{l.h.s. = \delta_H(e) \delta_L(l_{23456}) = |H| \delta_L(l_{23456})}. \tag{45}$$

>>> Proof of $2 \leftrightarrow 4$ Pachner move invariance

* *On the right hand side* of the move there is the integral

$$\int_G dg_{16} \int_{H^4} dh_{126}dh_{136}dh_{146}dh_{156} \int_L dl_{1236}dl_{1246}dl_{1256}dl_{1346}dl_{1356}dl_{1456} \left(\prod_{(jk\ell)\in M_2} \delta_G(g_{jk\ell})\right)\left(\prod_{(jk\ell m)\in M_3} \delta_H(h_{jk\ell m})\right)\left(\prod_{(jk\ell mn)\in M_4} \delta_L(l_{jk\ell mn})\right). \tag{46}$$

  * One integrates out g_{16} using $\delta_G(g_{126})$, h_{126} using $\delta_H(h_{1236})$, h_{136} using $\delta_H(h_{1346})$, and h_{146} using $\delta_H(h_{1456})$.
  * One integrates out l_{1236} using $\delta_L(l_{12346})$, l_{1246} using $\delta_L(l_{12456})$, l_{1346} using $\delta_L(l_{13456})$.
  * The remaining δ-functions on the group G reduces to $\delta_G(e)^3$,

$$\delta_G(g_{136}) = \delta_G(g_{146}) = \delta_G(g_{156}) = \delta_G(e).$$

  * One obtains that the remaining δ-functions on H reduce on $\delta_H(e)^3$,

$$\delta_H(h_{1256}) = \delta_H(h_{1356}) = \delta_H(h_{1456}) = \delta_H(e).$$

* For the remaining δ-function $\delta_L(l_{12356})$,

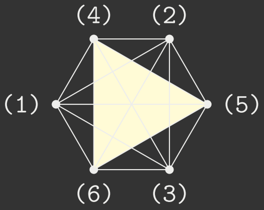
$$\delta_L(l_{12356}) = \delta_L\Big(l_{2456}{}^{-1} l_{2346}{}^{-1} l_{2356}(h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1235}(h_{256}g_{56} \rhd h_{125}{}^{-1}g_{56} \rhd h_{135}) \rhd' \Big((g_{35} \rhd h_{123}h_{356}^{-1}) \rhd' l_{3456})\{g_{56} \rhd h_{345}, (g_{56}g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{p}}^{-1}(g_{56} \rhd h_{345}(g_{56}g_{45}) \rhd (h_{123}h_{234}^{-1})h_{456}^{-1}) \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{234}\}_{\mathrm{p}}\Big)(h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1345} (h_{256}g_{56} \rhd h_{125}{}^{-1}g_{56} \rhd h_{145}) \rhd' ((g_{56}g_{45}) \rhd l_{1234})^{-1}(h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1245}^{-1}\Big). \tag{47}$$

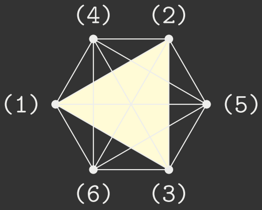
which is precisely the equation (44).
The remaining integration over the element h_{156} H and remaining integrations over the three elements l_{1246}, l_{1256}, and l_{1356}, are trivial, yielding the result of the r.h.s. to:

$$\boxed{r.h.s. = \delta_G(e)^3\,\delta_H(e)^3\,\delta_L(l_{12356}) = |G|^3\,|H|^3\,\delta_L(l_{12356})}. \tag{48}$$

The prefactors are $|G|^{-8}|H|^{-1}|L|^{-1}$ on the l.h.s., and $|G|^{-11}|H|^{-3}|L|^{-1}$ on the r.h.s. compensate for the left-over factors.

>>> $\mathbf{3 \leftrightarrow 3}$ Pachner move





$3 \leftrightarrow 3$

	l.h.s.	r.h.s
M_0		
M_1		
M_2	(456)	(123)
M_3	$(1456), (2456), (3456)$	$(1234), (1235), (1236)$
M_4	$(23456), (13456), (12456)$	$(12356), (12346), (12345)$.

>>> $\mathbf{3 \leftrightarrow 3}$ Pachner move

Left side

$$Z^{3\leftrightarrow 3}_{left} = \int_H dh_{456} \int_{L^3} dl_{1456} dl_{2456} dl_{3456} \delta_G(g_{456})\, \delta_H(h_{3456}) \delta_H(h_{2456}) \delta_H(h_{1456}) \delta_L(l_{23456}) \delta_L(l_{13456}) \delta_L(l_{12456}) Z_{\texttt{remainder}}\,, \tag{49}$$

Right side

$$Z^{3\leftrightarrow 3}_{right} = \int_H dh_{123} \int_{L^3} dl_{1234} dl_{1235} dl_{1236} \delta_G(g_{123})\, \delta_H(h_{1234}) \delta_H(h_{1235}) \delta_H(h_{1236}) \delta_L(l_{12356}) \delta_L(l_{12346}) \delta_L(l_{12345}) Z_{\texttt{remainder}}\,. \tag{50}$$

>>> Synopsis

* 2-crossed modules and 3-gauge theory
* Physically relevant models -The constrained $2BF$ actions describing the *Yang-Mills field* and *Einstein-Cartan gravity*, and constrained $3BF$ actions describing the *Klein-Gordon, Dirac, Weyl* and *Majorana fields* coupled to Yang-Mills fields and gravity in the standard way.
* Starting from the notion of Lie 3-groups, we generalize the integral picture of gauge theory to a 3-gauge theory that involves curves, surfaces, and volumes labeled with elements of non-Abelian groups.
* The definition of the discrete state sum model of topological higher gauge theory in dimension d=4.
* We prove that the state sum is well defined, i.e., invariant under the Pachner moves and thus independent of the chosen triangulation.

>>> Synopsis

* 2-crossed modules and 3-gauge theory
* Physically relevant models -The constrained $2BF$ actions describing the *Yang-Mills field* and *Einstein-Cartan gravity*, and constrained $3BF$ actions describing the *Klein-Gordon, Dirac, Weyl* and *Majorana fields* coupled to Yang-Mills fields and gravity in the standard way.
* Starting from the notion of Lie 3-groups, we generalize the integral picture of gauge theory to a 3-gauge theory that involves curves, surfaces, and volumes labeled with elements of non-Abelian groups.
* The definition of the discrete state sum model of topological higher gauge theory in dimension d=4.
* We prove that the state sum is well defined, i.e., invariant under the Pachner moves and thus independent of the chosen triangulation.

Thank you for your attention!