>>> Topological invariant of 4-manifolds based on a 3-group

>>> 2023 Workshop on Gravity, Holography, Strings and Noncommutative Geometry, Belgrade, Serbia

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Science Fund of the Republic of Serbia

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>>> A sketch of the talk

- 3-group and 3-gauge theory
  - ↪ based on *R. Picken and J. Faria Martins*, Diff. Geom. Appl. 29, 179 (2011), arXiv:0907.2566.
- $3BF$ action
  - ↪ Models with relevant dynamics *T. Radenković and M. Vojinović*, J. High Energy Phys.10, 222 (2019), arXiv:1904.07566.
- Quantization of the topological $3BF$ theory
  - ↪ the state sum Z is an example of Porter's TQFT for $d=4$ and $n=3$ *T. Porter*, J. Lond. Math. Soc. (2)58, No. 3, 723 (1998), MR 1678163.
- Pachner move invariance
  - ↪ The construction of the state sum Z and a proof that the $3BF$ state sum is invariant under Pachner moves.
    T. Radenković and M. Vojinović, arXiv: 2201.02572.
  - ↪ This is a generalization of the state sum based on the classical $2BF$ action with the underlying 2-group structure
    F. Girelli, H. Pfeiffer and E. M. Popescu, Jour. Math. Phys. 49, 032503 (2008), arXiv:0708.3051.
- Conclusions

2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G, \triangleright, \{\_,\_\}_{\mathrm{p}})$

* Groups G, H, and L;
* maps ∂ and δ ($\partial\delta = 1_G$);
* an action $\triangleright$ of the group G on all three groups;
* a map $\{\_\,,\_\}_{\mathrm{p}}$ called the *Peiffer lifting*:

$$\{\_\,,\_\}_{\mathrm{p}} : H \times H \to L\,.$$

Certain axioms hold true among all these maps:

1. $\delta(\{h_1, h_2\}_{\mathrm{p}}) = \langle h_1\,, h_2\rangle_{\mathrm{p}}\,, \qquad \forall h_1, h_2 \in H\,,$
2. $[l_1, l_2] = \{\delta(l_1)\,, \delta(l_2)\}_{\mathrm{p}}\,, \qquad \forall l_1\,, l_2 \in L$. Here, the notation $[l,k] = lkl^{-1}k^{-1}$ is used;
3. $\{h_1 h_2, h_3\}_{\mathrm{p}} = \{h_1, h_2 h_3 h_2^{-1}\}_{\mathrm{p}} \partial(h_1) \triangleright \{h_2, h_3\}_{\mathrm{p}}\,, \qquad \forall h_1, h_2, h_3 \in H$;
4. $\{h_1, h_2 h_3\}_{\mathrm{p}} = \{h_1, h_2\}_{\mathrm{p}} \{h_1, h_3\}_{\mathrm{p}} \{\langle h_1, h_3\rangle_{\mathrm{p}}^{-1}, \partial(h_1) \triangleright h_2\}_{\mathrm{p}}\,, \qquad \forall h_1, h_2, h_3 \in H$;
5. $\{\delta(l), h\}_{\mathrm{p}} \{h, \delta(l)\}_{\mathrm{p}} = l(\partial(h) \triangleright l^{-1})\,, \qquad \forall h \in H\,, \quad \forall l \in L\,.$

>>> The $3BF$ theory

One can now generalize the notion of parallel transport from curves to surfaces and volumes.

* Given a 2-crossed module, one can define a *3-connection*, an ordered triple (α,β,γ), where α, β, and γ are algebra-valued differential forms,

$$\begin{aligned} &\alpha = \alpha^{\alpha}{}_{\mu}\,\tau_{\alpha}\,\mathrm{d}x^{\mu}\,, && \alpha \in \mathcal{A}^1(\mathcal{M}_4,\mathfrak{g})\,, \\ &\beta = \beta^{a}{}_{\mu\nu}\,t_a\,\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\,, && \beta \in \mathcal{A}^2(\mathcal{M}_4,\mathfrak{h})\,, \\ &\gamma = \gamma^{A}{}_{\mu\nu\rho}\,T_A\,\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\rho}\,, && \gamma \in \mathcal{A}^3(\mathcal{M}_4,\mathfrak{l})\,. \end{aligned} \tag{1}$$

* Then introduce the *line, surface and volume holonomies*,

$$g = \mathcal{P}\exp\int_{\gamma}\alpha\,,\quad h = \mathcal{P}\exp\int_{S}\beta\,,\quad l = \mathcal{P}\exp\int_{V}\gamma\,. \tag{2}$$

* The corresponding *fake 3-curvature* $(\mathcal{F},\mathcal{G},\mathcal{H})$ is defined as:

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,,\qquad \mathcal{G} = \mathrm{d}\beta + \alpha\wedge^{\triangleright}\beta - \delta\gamma\,, \\ \mathcal{H} = \mathrm{d}\gamma + \alpha\wedge^{\triangleright}\gamma + \{\beta\wedge\beta\}_{\mathrm{pf}}\,. \tag{3}$$

>>> The $3BF$ theory

At this point one can construct the so-called $3BF$ theory.

* For a manifold $\mathcal{M}_4$ and the 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \rhd\,, \{\_\,,\_\}_{\mathrm{pf}})$, that gives rise to 3-curvature $(\mathcal{F},\mathcal{G},\mathcal{H})$, one defines the $3BF$ action as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,. \tag{4}$$

  * $3BF$ theory is a topological gauge theory,
  * it is based on the 3-group structure,
  * it is a generalization of an ordinary BF theory for a given Lie group G.
  * Physically relevant models

    The constrained $2BF$ actions for
    * *Yang-Mills field,*
    * and *Einstein-Cartan gravity,*

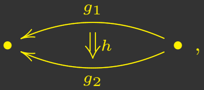
    and constrained $3BF$ actions describing
    * *Klein-Gordon field,*
    * *Dirac field,*
    * *Weyl fields,*
    * and *Majorana fields*

    coupled gravity in the standard way are formulated.

>>> 3-gauge theory

* Curves are labeled with the elements of G, and the elements are composed as

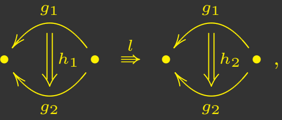
* Surfaces are labeled with the elements $h \in H$. We split the boundary into two curves, the source curve $g_1 \in G$ and the target curve $g_2 \in G$,



so that the surface $h \in H$ satisfies:

$$\partial(h) = g_2 g_1^{-1} \, .$$

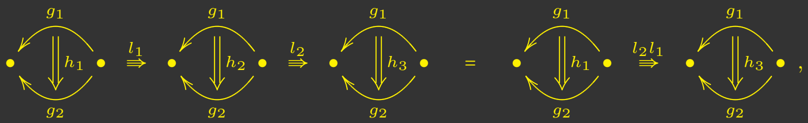
* Volumes are labeled with the elements $l \in L$. We split the boundary into the source surface $\partial_3^-(l) = h_1$ and the target surface $\partial_3^+(l) = h_2$, and the common boundary of h_1 and h_2 we split into the source curve $\partial_2^-(l) = g_1$ and the target curve $\partial_2^+(l) = g_2$,



$$\delta(l) = h_2 h_1^{-1} \, .$$

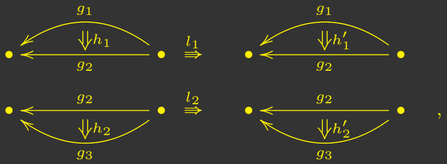
>>> 3-gauge theory

* *Upward composition*. The upward composition of 3-morphisms (g_1, h_1, l_1) and (g_1, h_2, l_2), when they are compatible, when $\partial_3^+(l_1) = \partial_3^-(l_2)$,



$$(g_1, h_2, l_2) \#_3 (g_1, h_1, l_1) = (g_1, h_1, l_2 l_1)\,. \tag{5}$$

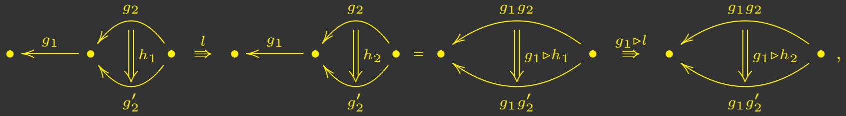
* *Vertical composition*. The vertical composition of two 3-morphisms (g_1, h_1, l_1) and (g_2, h_2, l_2), when they are compatible, when $\partial_2^+(l_1) = \partial_2^-(l_2)$,



$$(g_2, h_2, l_2) \#_2 (g_1, h_1, l_1) = (g_1, h_2 h_1, l_2 (h_2 \rhd' l_1))\,. \tag{6}$$

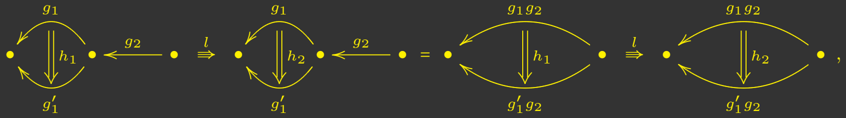
>>> 3-gauge theory

* *Whiskering of the 3-morphisms with morphisms*. Whiskering of a 3-morphism by a morphism from the left is the composition of a volume $l \in L$ and curve $g_1 \in G$ from the left, when they are compatible, when $\partial_1^+(l) = \partial_1^-(g_1)$,



$$g_1 \#_1 (g_2, h_1, l) = (g_1 g_2, g_1 \triangleright h, g_1 \triangleright l)\,. \tag{7}$$

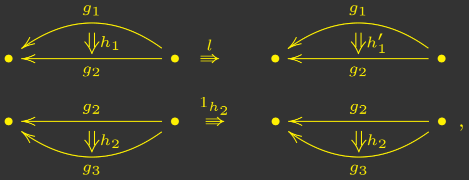
One can whisker a 3-morphism by a morphism from the right, when they are compatible, $\partial_1^-(l) = \partial_1^+(g_2)$,



$$(g_1, h_1, l) \#_1 g_2 = (g_1 g_2, h_1, l)\,. \tag{8}$$

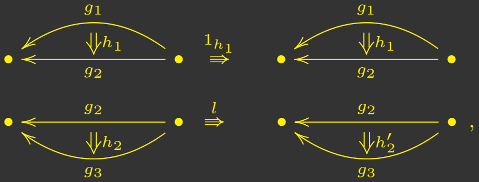
>>> 3-gauge theory

* *Whiskering of 3-morphisms with 2-morphisms*. Whiskering of a 3-morphism with a 2-morphisms from below, when they are compatible, $\partial_2^+(l) = \partial_2^-(h_2)$,



$$(g_1, h_1, l)\#_2(g_2, h_2) = (g_1, h_2 h_1, h_2 \triangleright' l)\,. \tag{9}$$

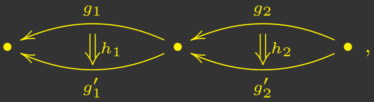
* Whiskering a 3-morphism by 2-morphism from above, when they are compatible, when $\partial_2^-(l) = \partial_2^+(h_1)$,



$$(g_1, h_1)\#_2(g_2, h_2, l) = (g_1, h_2 h_1, l)\,. \tag{10}$$

>>> 3-gauge theory

* *The interchanging 3-arrow.* The horizontal composition of two 2-morphisms h_1 and h_2, when they are compatible, when $\partial_1^-(h_1) = \partial_1^+(h_2)$,



that results in a 3-morphism l, with source surface and target surfaces

$$\partial_3^-(l) = \big((g_1,h_1)\#_1 g_2'\big)\#_2\big(g_1\#_1(g_2,h_2)\big)\,, \quad \partial_3^+(l) = \big(g_1'\#_1(g_2,h_2)\big)\#_2\big((g_1,h_1)\#_1 g_2\big)\,.$$

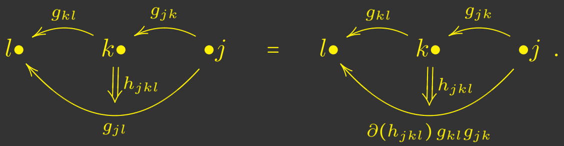
One obtains,

$$(g_1,h_1)\#_1(g_2,h_2) = \left(g_1g_2, h_1 g_1 \triangleright h_2, \{h_1, g_1 \triangleright h_2\}_{\mathrm{p}}^{-1}\right)\,. \tag{11}$$

>>> 3-gauge theory

Lemma

Let us consider a triangle, $(jk\ell)$. The edges $(jk), j<k$, are labeled by group elements $g_{jk}\in G$ and the triangle $(jk\ell), j<k<\ell$, by element $h_{jk\ell}\in H$.



(12)

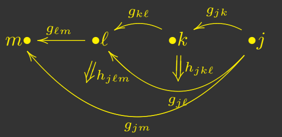
The curve $\gamma_1 = g_{k\ell}g_{jk}$ is the source and the curve $\gamma_2 = g_{j\ell}$ is the target of the surface morphism $\Sigma:\gamma_1\to\gamma_2$, labeled by the group element $h_{jk\ell}$,

$$g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk}\,. \tag{13}$$

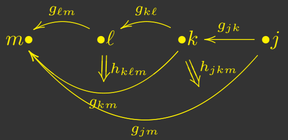
>>> 3-gauge theory

Lemma

Let us consider a tetrahedron, $(jk\ell m)$.



$$= (g_{\ell m} g_{j\ell}, h_{j\ell m}) \#_2 \big(g_{\ell m} \#_1 (g_{k\ell} g_{jk}, h_{jk\ell})\big) = \big(g_{\ell m} g_{k\ell} g_{jk}, h_{j\ell m}(g_{\ell m} \triangleright h_{jk\ell})\big) . \tag{14}$$



$$= (g_{km} g_{jk}, h_{jkm}) \#_2 \big((g_{\ell m} g_{k\ell}, h_{k\ell m}) \#_1 g_{jk}\big) = (g_{\ell m} g_{k\ell} g_{jk}, h_{jkm} h_{k\ell m}) . \tag{15}$$

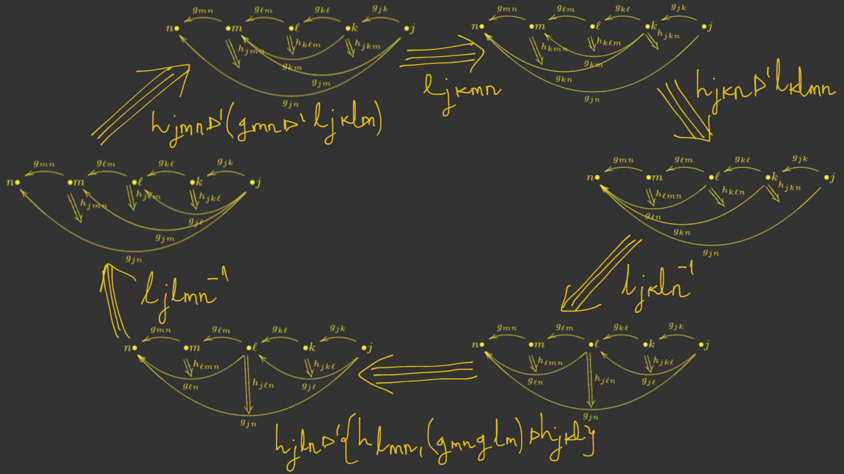
Moving from surface shown on the diagram (14) to the surface shown on the diagram (15) is determined by the group element $l_{jk\ell m}$,

$$h_{jkm} h_{k\ell m} = \delta(l_{jk\ell m}) h_{j\ell m} (g_{\ell m} \triangleright h_{jk\ell}) . \tag{16}$$

>>> 3-gauge theory

Lemma (δ_L)

We consider a 4-simplex, $(jk\ell mn)$. We cut the 4-simplex volume along the surface $h_{jmn} g_{mn} \triangleright (h_{j\ell m} g_{\ell m} \triangleright h_{jk\ell})$.



Lemma (δ_L)

After the upward composition of these 3-morphisms, the obtained 3-morphism is the identity morphism with source and target surface $\mathcal{V}_1 = \mathcal{V}_2 = h_{jmn} g_{mn} \rhd (h_{j\ell m} g_{\ell m} \rhd h_{jk\ell})$,

$$l^{-1}_{j\ell mn} \, h_{j\ell n} \rhd' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \rhd h_{jk\ell}\}_{\mathrm{p}} \, l^{-1}_{jk\ell n} (h_{jkn} \rhd' l_{k\ell mn}) l_{jkmn} h_{jmn} \rhd' (g_{mn} \rhd l_{jk\ell m}) = e \,. \tag{17}$$

>>> Quantization of the topological $3BF$ theory

We want to construct a *state sum model* from the classical S_{3BF} action by the usual *spinfoam quantization procedure*.

$$Z = \int \mathcal{D}\alpha\, \mathcal{D}\beta\, \mathcal{D}\gamma\, \mathcal{D}B\, \mathcal{D}C\, \mathcal{D}D\, \exp\left(i \int_{M_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \right). \tag{18}$$

↪ The formal integration over the Lagrange multipliers B, C, and D leads to:

$$Z = \mathcal{N} \int \mathcal{D}\alpha\, \mathcal{D}\beta\, \mathcal{D}\gamma\;\; \delta(\mathcal{F})\delta(\mathcal{G})\delta(\mathcal{H}). \tag{19}$$

↪ Discretization of the 3-connection:

- $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g}) \mapsto g_\epsilon \in G$ coloring the edges $\epsilon = (jk) \in \Lambda_1$,
- $\beta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h}) \mapsto h_\Delta \in H$ coloring the triangles $\Delta = (jk\ell) \in \Lambda_2$,
- $\gamma \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{l}) \mapsto l_\tau \in L$ coloring the tetrahedrons $\tau = (jk\ell m) \in \Lambda_3$.

$$\left.\begin{array}{lcl} \int \mathcal{D}\alpha & \mapsto & \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \\ \int \mathcal{D}\beta & \mapsto & \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \\ \int \mathcal{D}\gamma & \mapsto & \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \end{array}\right\} \longrightarrow$$

The disretization of path integral measures.

>>> Quantization of the toplogical $3BF$ theory

↪ The condition $\delta(\mathcal{F})$ is disretized as

$$\delta(\mathcal{F}) = \prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\,, \qquad \delta_G(g_{jk\ell}) = \delta_G\Big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\Big)\,. \tag{20}$$

↪ The condition $\delta(\mathcal{G})$ on the fake curvature 3-form reads

$$\delta(\mathcal{G}) = \prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\,, \tag{21}$$

$$\delta_H(h_{jk\ell m}) = \delta_H\Big(\delta(l_{jk\ell m}) h_{j\ell m}\,(g_{\ell m} \triangleright h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\Big)\,. \tag{22}$$

↪ The condition $\delta(\mathcal{H})$ is disretized as

$$\delta(\mathcal{H}) = \prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\,, \tag{23}$$

$$\delta_L(l_{jk\ell mn}) = \delta_L\Big(l_{j\ell mn}^{-1}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}}\, l_{jk\ell n}^{-1} (h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})\Big)\,. \tag{24}$$

...all off this $\Longrightarrow$

$$Z = \mathcal{N} \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \Bigg(\prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\Bigg)\Bigg(\prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\Bigg)\Bigg(\prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\Bigg). \tag{25}$$

This expression can be made independent of the triangulation if one appropriately chooses the constant factor $\mathcal{N}$.

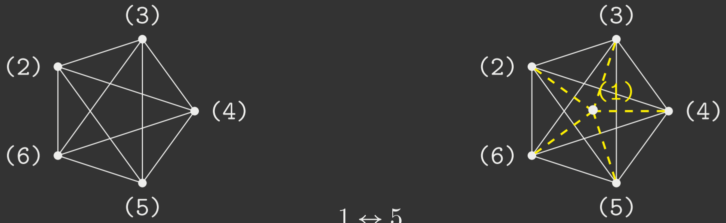
Definition

Let $\mathcal{M}_4$ be a compact and oriented combinatorial 4-manifold, and $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G, \triangleright, \{\_,\_\}_{\rm pf})$ be a 2-crossed module. The state sum of *topological higher gauge theory* is defined by

$$
\begin{aligned}
Z \;=\; & |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\,|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|} \\
& \times\left(\prod_{(jk)\in\Lambda_1}\int_G dg_{jk}\right)\left(\prod_{(jk\ell)\in\Lambda_2}\int_H dh_{jk\ell}\right)\left(\prod_{(jk\ell m)\in\Lambda_3}\int_L dl_{jk\ell m}\right) \\
& \times\left(\prod_{(jk\ell)\in\Lambda_2}\delta_G\big(\partial(h_{jk\ell})\,g_{k\ell}\,g_{jk}\,g_{j\ell}^{-1}\big)\right)\left(\prod_{(jk\ell m)\in\Lambda_3}\delta_H\big(\delta(l_{jk\ell m})h_{j\ell m}\,(g_{\ell m}\triangleright h_{jk\ell})\,h_{k\ell m}^{-1}\,h_{jkm}^{-1}\big)\right) \\
& \times\left(\prod_{(jk\ell mn)\in\Lambda_4}\delta_L\big(l_{j\ell mn}^{-1}\,h_{j\ell n}\triangleright'\{h_{\ell mn},(g_{mn}g_{\ell m})\triangleright h_{jk\ell}\}_{\rm p}\,l_{jk\ell n}^{-1}(h_{jkn}\triangleright' l_{k\ell mn})l_{jkmn}h_{jmn}\triangleright'(g_{mn}\triangleright l_{jk\ell m})\big)\right).
\end{aligned}
\tag{26}
$$

Here $|\Lambda_0|$ denotes the number of vertices, $|\Lambda_1|$ edges, $|\Lambda_2|$ triangles, $|\Lambda_3|$ tetrahedrons, and $|\Lambda_4|$ 4-simplices of the triangulation.

>>> $1 \leftrightarrow 5$ Pachner move



$1 \leftrightarrow 5$

Right side

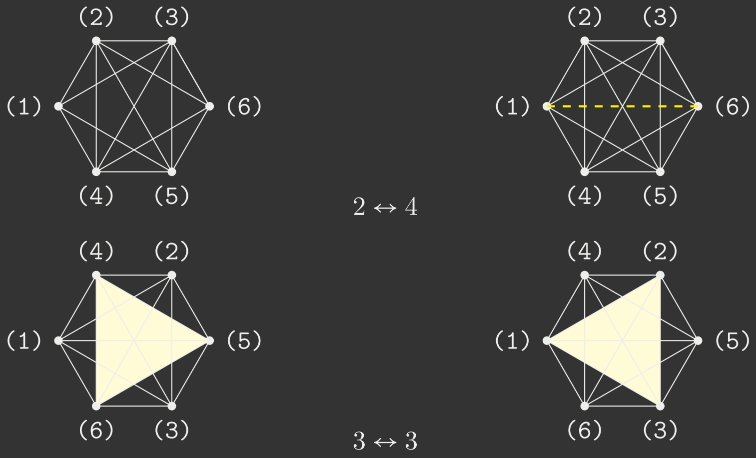
$$Z^{1\leftrightarrow 5}_{\texttt{right}} = |G|^{-11}|H|^{-4}|L|^{-1} \int_{G^5} \prod_{(jk)\in M_1} dg_{jk} \int_{H^{10}} \prod_{(jk\ell)\in M_2} dh_{jk\ell} \int_{L^{10}} \prod_{(jklm)\in M_3} dl_{jklm} \cdot \left(\prod_{(jk\ell)\in M_2} \delta_G(g_{jk\ell}) \right) \left(\prod_{(jk\ell m)\in M_3} \delta_H(h_{jk\ell m}) \right) \left(\prod_{(jk\ell mn)\in M_4} \delta_L(l_{jk\ell mn}) \right) Z_{\texttt{remainder}} , \tag{27}$$

Left side

$$Z^{1\leftrightarrow 5}_{\texttt{left}} = |G|^{-5}|H|^{0}|L|^{-1}\delta_L(l_{23456}) Z_{\texttt{remainder}} . \tag{28}$$

The $Z_{\texttt{remainder}}$ denotes the part of the state sum that is the same on both sides of the move, and thus irrelevant for the proof of invariance.

>>> $\mathbf{2} \leftrightarrow \mathbf{4}$ and $\mathbf{3} \leftrightarrow \mathbf{3}$ Pachner moves



>>> Synopsis

* 2-crossed modules and 3-gauge theory
* Physically relevant models -The constrained $2BF$ actions describing the *Yang-Mills field* and *Einstein-Cartan gravity*, and constrained $3BF$ actions describing the *Klein-Gordon, Dirac, Weyl* and *Majorana fields* coupled to Yang-Mills fields and gravity in the standard way.
* Starting from the notion of Lie 3-groups, we generalize the integral picture of gauge theory to a 3-gauge theory that involves curves, surfaces, and volumes labeled with elements of non-Abelian groups.
* The definition of the discrete state sum model of topological higher gauge theory in dimension d=4.
* We prove that the state sum is well defined, i.e., invariant under the Pachner moves and thus independent of the chosen triangulation.

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Thank you for your attention!