

# Rational $so(3)$ Gaudin model - a comprehensive treatment

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# Talk outline

- A few words on motivation
- A few words on Gaudin model in general
- $SO(3)$  Gaudin model with boundary
  - Preliminaries (r-matrix, Lax operator)
  - Nontrivial “boundary” conditions
  - Gaudin Hamiltonians
  - General boundary  $\rightarrow$  special vacuum
  - Solving ABA
- Knizhnik-Zamolodchikov equations
- Norm and scalar product formulas

# Motivation?

- So few problems that we can exactly solve
  - Any such is precious
  - The more general, the better
  - Expected to provide insights to realistic problems
  - Gaudin model – integrable, with long range interactions
- Talk based on...
  - N. Manojlović, I. Salom, Nuclear Physics B, 978, 2022

# Gaudin model

- Introduced as a “quasi-classical” expansion of the spin-chain models, for example:

$$R(\lambda) = \lambda 1 + \eta \mathcal{P} = \begin{pmatrix} \lambda + \eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda + \eta \end{pmatrix}$$

Yang-Baxter

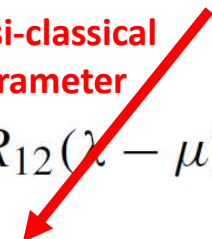


$$\frac{1}{\lambda} R(\lambda) = 1 - \eta r(\lambda)$$

classical  
r-matrix

quasi-classical  
parameter

“Classical”  
Yang-Baxter



$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu)$$



$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0$$

# Gaudin model - independent approach

- Classical r-matrix:

$$[r_{32}(\lambda_3, \lambda_2), r_{13}(\lambda_1, \lambda_3)] + [r_{12}(\lambda_1, \lambda_2), r_{13}(\lambda_1, \lambda_3) + r_{23}(\lambda_2, \lambda_3)] = 0$$

- Lax operator satisfies:

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = [r_{00'}(\lambda, \mu), \mathcal{L}_0(\lambda)] - [r_{0'0}(\mu, \lambda), \mathcal{L}_{0'}(\mu)]$$

- Then, if we define generating function:

$$\tau(\lambda) = \text{tr}_0 \mathcal{L}_0^2(\lambda)$$

- It will commute for different values of “spectral parameters”:

$$[\tau(\lambda), \tau(\mu)] = 0$$

# So what?

- For a given r-matrix, we can find “local” realization of Lax operator that acts in a multiparticle Hilbert space  $\mathcal{H} = \bigotimes_{m=1}^N V_m$
- Generating function, or its derivatives (residues) can take form of Hamiltonians with nontrivial interactions
- Solving eigenproblem of the generating function  $\tau(\lambda)$  provides all relevant information about system (energies, Hamiltonian eigenstates, integrals of motion)
- There is an efficient way to solve  $\tau(\lambda)$  eigenproblem, know as Algebraic Bethe Ansatz (ABA)
- Gaudin model exhibits long-range interactions

# Rational so(3) Gaudin model

- We start from the classical r-matrix:

$$r_{12}(\lambda) = -\frac{\vec{S}_1 \cdot \vec{S}_2}{\lambda} \quad S^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- It corresponds to Lax operator:

$$L_0(\lambda) = \sum_{m=1}^N \frac{\vec{S}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} = \sum_{m=1}^N \frac{1}{\lambda - \alpha_m} \left( S_0^3 \otimes S_m^3 + \frac{1}{2} (S_0^+ \otimes S_m^- + S_0^- \otimes S_m^+) \right)$$

- where spin operators  $S_m^\alpha, \alpha = +, -, 3$  live in  $\mathcal{H} = \bigotimes_{m=1}^N V_m = (\mathbb{C}^{2s+1})^{\otimes N}$

and satisfy:  $[S_m^3, S_n^\pm] = \pm S_m^\pm \delta_{mn}, \quad [S_m^+, S_n^-] = 2S_m^3 \delta_{mn}$

- Tightly related to rational SO(3) Heisenberg spin-chain model. Lax operator there has form:

$$\mathbb{L}_{0m}(\lambda) = \mathbb{1} + \frac{2\eta}{\lambda} \vec{S}_0 \cdot \vec{S}_m + \frac{2\eta^2}{\lambda(\lambda + \eta)} \left( (\vec{S}_0 \cdot \vec{S}_m)^2 - \mathbb{1} \right)$$

# Rational so(3) Gaudin model

- It holds:  $[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0, \quad r_{21}(-\lambda) = -r_{12}(\lambda)$   
 $[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)]$

- So that, if we define:  $\tau(\lambda) = \frac{1}{2} \text{tr} L^2(\lambda)$

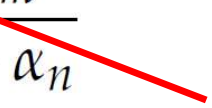
- it will commute for different values of the spectral parameter:

$$[\tau(\lambda), \tau(\mu)] = 0$$

- It will also commute with its residue, that has the form of a Hamiltonian:

$$\text{Res}_{\lambda=\alpha_m} \tau(\lambda) = 4 H_m$$

$$H_m = \sum_{n \neq m}^N \frac{\vec{S}_n \vec{S}_m}{\alpha_m - \alpha_n}$$

 long-range interaction



# Introducing nontrivial “boundary”

- Name comes from the relation to the boundary terms of a spin-chain
- Model can be generalized by a K matrix satisfying “reflection equation”:

$$\begin{aligned} r_{12}(\lambda - \mu)K_1(\lambda)K_2(\mu) + K_1(\lambda)r_{21}(\lambda + \mu)K_2(\mu) = \\ = K_2(\mu)r_{12}(\lambda + \mu)K_1(\lambda) + K_2(\mu)K_1(\lambda)r_{21}(\lambda - \mu). \end{aligned}$$

- SO(3) solutions can be obtained using “fusion procedure”:

$$K(\lambda) = \begin{pmatrix} (\xi - \nu\lambda)^2 & -\sqrt{2}\psi\lambda(\xi - \nu\lambda) & \psi^2\lambda^2 \\ -\sqrt{2}\varphi\lambda(\xi - \nu\lambda) & \xi^2 + (\psi\varphi - \nu^2)\lambda^2 & -\sqrt{2}\psi\lambda(\xi + \nu\lambda) \\ \varphi^2\lambda^2 & -\sqrt{2}\varphi\lambda(\xi + \nu\lambda) & (\xi + \nu\lambda)^2 \end{pmatrix}$$

- Commonly, we fix some of the parameters to zero, to have ground state for algebraic Bethe ansatz
- This time we do better!

# Nontrivial “boundary”

- Now, we can define more general:

$$r_{00'}^K(\lambda, \mu) = r_{00'}(\lambda - \mu) - K_{0'}(\mu)r_{00'}(\lambda + \mu)K_{0'}^{-1}(\mu)$$

$$\mathcal{L}_0(\lambda) = L_0(\lambda) - K_0(\lambda)L_0(-\lambda)K_0^{-1}(\lambda)$$

- that again satisfy:

$$[r_{32}^K(\lambda_3, \lambda_2), r_{13}^K(\lambda_1, \lambda_3)] + [r_{12}^K(\lambda_1, \lambda_2), r_{13}^K(\lambda_1, \lambda_3) + r_{23}^K(\lambda_2, \lambda_3)] = 0$$

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = [r_{00'}^K(\lambda, \mu), \mathcal{L}_0(\lambda)] - [r_{0'0}^K(\mu, \lambda), \mathcal{L}_{0'}(\mu)]$$

- so we define more general generating function

$$\tau(\lambda) = \text{tr}_0 \mathcal{L}_0^2(\lambda) \quad \rightarrow \quad [\tau(\lambda), \tau(\mu)] = 0$$

# Gaudin Hamiltonians

- We take residues of generating function:

$$\operatorname{Res}_{\lambda=\alpha_m} \tau(\lambda) = 4 H_m$$

- Result has a form of “tunable” many-particle interacting Hamiltonian-like operators:

**By solving eigenproblem of generating function we also find energies and H. eigenstates!**

$$\begin{aligned}
 H_m = (\pm) \operatorname{Res}_{\lambda=\pm\alpha_m} \tau(\lambda) = & \frac{1}{\xi^2} \left( -\frac{\alpha_m}{2} (\psi^2 (S_m^+)^2 + \varphi^2 (S_m^-)^2) \right. \\
 & + 2\psi\nu (S_m^+ S_m^3 + S_m^3 S_m^+) + 2\varphi\nu (S_m^- S_m^3 + S_m^3 S_m^-) + \frac{\xi^2 + \nu^2 \alpha_m^2}{2\alpha_m} (S_m^+ S_m^- + S_m^- S_m^+) \Big) \\
 & + \frac{\alpha_m}{\xi^2 - (\psi\varphi + \nu^2) \alpha_m^2} \sum_{n \neq m}^N \left( \frac{4(\xi^2 - \psi\varphi \alpha_m \alpha_n - \nu^2 \alpha_m^2)}{\alpha_m^2 - \alpha_n^2} S_m^3 S_n^3 - \frac{\alpha_m}{\alpha_m + \alpha_n} (\psi^2 S_m^+ S_n^+ + \varphi^2 S_m^- S_n^-) \right. \\
 & + 2\psi\nu (S_m^+ S_n^3 + S_m^3 S_n^+) + 2\varphi\nu (S_m^- S_n^3 + S_m^3 S_n^-) + \frac{2(\xi^2 - (\psi\varphi + \nu^2) \alpha_m \alpha_n) - \psi\varphi \alpha_m (\alpha_m - \alpha_n)}{\alpha_m^2 - \alpha_n^2} \times \\
 & \left. \times (S_m^- S_n^+ + S_m^+ S_n^-) \right) \\
 & + \frac{\xi \cdot \alpha_m}{\xi^2 - (\psi\varphi + \nu^2) \alpha_m^2} \sum_{n \neq m}^N \frac{1}{\alpha_m + \alpha_n} (2\psi (S_m^+ S_n^3 - S_m^3 S_n^+) + 2\nu (S_m^- S_n^+ - S_m^+ S_n^-) + 2\varphi (S_m^3 S_n^- - S_m^- S_n^3))
 \end{aligned}$$

long-range interaction

“boundary” parameters

# Algebraic Bethe Ansatz

- Writing:  $\mathcal{L}_0(\lambda) = L_0(\lambda) - K_0(\lambda)L_0(-\lambda)K_0^{-1}(\lambda) = \begin{pmatrix} H(\lambda) & \frac{1}{\sqrt{2}}F(\lambda) & 0 \\ \frac{1}{\sqrt{2}}E(\lambda) & 0 & \frac{1}{\sqrt{2}}F(\lambda) \\ 0 & \frac{1}{\sqrt{2}}E(\lambda) & -H(\lambda) \end{pmatrix}$

- from:  $[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = [r_{00'}(\lambda, \mu), \mathcal{L}_0(\lambda)] - [r_{0'0}(\mu, \lambda), \mathcal{L}_{0'}(\mu)]$

- we obtain:

$$[E(\lambda), E(\mu)] = \frac{-2\varphi^2}{\lambda + \mu} \left( \frac{\mu^2}{\xi^2 - (\psi\varphi + v^2)\mu^2} H(\lambda) - \frac{\lambda^2}{\xi^2 - (\psi\varphi + v^2)\lambda^2} H(\mu) \right),$$

$$+ \frac{2\varphi}{\lambda + \mu} \left( \frac{(\xi + v\mu)\mu}{\xi^2 - (\psi\varphi + v^2)\mu^2} E(\lambda) - \frac{(\xi + v\lambda)\lambda}{\xi^2 - (\psi\varphi + v^2)\lambda^2} E(\mu) \right),$$

$$[F(\lambda), F(\mu)] = \frac{2\psi^2}{\lambda + \mu} \left( \frac{\mu^2}{\xi^2 - (\psi\varphi + v^2)\mu^2} H(\lambda) - \frac{\lambda^2}{\xi^2 - (\psi\varphi + v^2)\lambda^2} H(\mu) \right)$$

$$+ \frac{2\psi}{\lambda + \mu} \left( \frac{(\xi - v\mu)\mu}{\xi^2 - (\psi\varphi + v^2)\mu^2} F(\lambda) - \frac{(\xi - v\lambda)\lambda}{\xi^2 - (\psi\varphi + v^2)\lambda^2} F(\mu) \right),$$

$$[H(\lambda), H(\mu)] = \frac{-\psi}{\lambda + \mu} \left( \frac{(\xi + v\mu)\mu}{\xi^2 - (\psi\varphi + v^2)\mu^2} E(\lambda) - \frac{(\xi + v\lambda)\lambda}{\xi^2 - (\psi\varphi + v^2)\lambda^2} E(\mu) \right)$$

$$+ \frac{-\varphi}{\lambda + \mu} \left( \frac{(\xi - v\mu)\mu}{\xi^2 - (\psi\varphi + v^2)\mu^2} F(\lambda) - \frac{(\xi - v\lambda)\lambda}{\xi^2 - (\psi\varphi + v^2)\lambda^2} F(\mu) \right),$$

$$[H(\lambda), E(\mu)] = \frac{\varphi}{\lambda + \mu} \left( \frac{\varphi\mu^2}{\xi^2 - (\psi\varphi + v^2)\mu^2} F(\lambda) - \frac{2(\xi - v\lambda)\lambda}{\xi^2 - (\psi\varphi + v^2)\lambda^2} H(\mu) \right)$$

$$- \frac{1}{(\lambda - \mu)(\lambda + \mu)} \left( \frac{(2(\xi - v\lambda)(\xi + v\mu) - \psi\varphi(\lambda + \mu)\mu)}{\xi^2 - (\psi\varphi + v^2)\mu^2} E(\lambda) \right.$$

$$\left. - \frac{2(\xi^2 - (\psi\varphi\mu + v^2\lambda)\lambda)}{\xi^2 - (\psi\varphi + v^2)\lambda^2} E(\mu) \right),$$

$$[H(\lambda), F(\mu)] = \frac{-\psi}{\lambda + \mu} \left( \frac{\psi\mu^2}{\xi^2 - (\psi\varphi + v^2)\mu^2} E(\lambda) + \frac{2(\xi + v\lambda)\lambda}{\xi^2 - (\psi\varphi + v^2)\lambda^2} H(\mu) \right)$$

$$+ \frac{1}{(\lambda - \mu)(\lambda + \mu)} \left( \frac{(2(\xi - v\mu)(\xi + v\lambda) - \psi\varphi(\lambda + \mu)\mu)}{\xi^2 - (\psi\varphi + v^2)\mu^2} F(\lambda) \right.$$

$$\left. - \frac{2(\xi^2 - (\psi\varphi\mu + v^2\lambda)\lambda)}{\xi^2 - (\psi\varphi + v^2)\lambda^2} F(\mu) \right),$$

$$[F(\lambda), E(\mu)] = \frac{-2}{\lambda + \mu} \left( \frac{\varphi(\xi + v\mu)\mu}{\xi^2 - (\psi\varphi + v^2)\mu^2} F(\lambda) - \frac{\psi(\xi - v\lambda)\lambda}{\xi^2 - (\psi\varphi + v^2)\lambda^2} E(\mu) \right)$$

$$+ \frac{2}{(\lambda - \mu)(\lambda + \mu)} \left( \frac{(2(\xi - v\lambda)(\xi + v\mu) - \psi\varphi(\lambda + \mu)\mu)}{\xi^2 - (\psi\varphi + v^2)\mu^2} H(\lambda) \right.$$

$$\left. - \frac{(2(\xi - v\lambda)(\xi + v\mu) - \psi\varphi(\lambda + \mu)\lambda)}{\xi^2 - (\psi\varphi + v^2)\lambda^2} H(\mu) \right).$$

**Terrible!?!**

# Wise change of basis

- Define new operators:

$$\mathcal{E}(\lambda) = \frac{1}{2\psi\sqrt{\psi\varphi + \nu^2}} \left( \psi^2 E(\lambda) - \left( \psi\varphi + 2\nu \left( \nu - \sqrt{\psi\varphi + \nu^2} \right) \right) F(\lambda) + 2\psi \left( \nu - \sqrt{\psi\varphi + \nu^2} \right) H(\lambda) \right)$$

$$\mathcal{F}(\lambda) = \frac{1}{2\psi\sqrt{\psi\varphi + \nu^2}} \left( -\psi^2 E(\lambda) + \left( \psi\varphi + 2\nu \left( \nu + \sqrt{\psi\varphi + \nu^2} \right) \right) F(\lambda) - 2\psi \left( \nu + \sqrt{\psi\varphi + \nu^2} \right) H(\lambda) \right)$$

$$\mathcal{H}(\lambda) = \frac{1}{2\sqrt{\psi\varphi + \nu^2}} (\psi E(\lambda) + \varphi F(\lambda) + 2\nu H(\lambda))$$

- Now:  $[\mathcal{E}(\lambda), \mathcal{E}(\mu)] = [\mathcal{F}(\lambda), \mathcal{F}(\mu)] = [\mathcal{H}(\lambda), \mathcal{H}(\mu)] = 0,$

$$[\mathcal{H}(\lambda), \mathcal{E}(\mu)] = \frac{-2}{\lambda^2 - \mu^2} \left( \mu \frac{\xi - \lambda\sqrt{\psi\varphi + \nu^2}}{\xi - \mu\sqrt{\psi\varphi + \nu^2}} \mathcal{E}(\lambda) - \lambda \mathcal{E}(\mu) \right),$$

$$[\mathcal{H}(\lambda), \mathcal{F}(\mu)] = \frac{2}{\lambda^2 - \mu^2} \left( \mu \frac{\xi + \lambda\sqrt{\psi\varphi + \nu^2}}{\xi + \mu\sqrt{\psi\varphi + \nu^2}} \mathcal{F}(\lambda) - \lambda \mathcal{F}(\mu) \right),$$

$$[\mathcal{F}(\lambda), \mathcal{E}(\mu)] = \frac{4}{\lambda^2 - \mu^2} \left( \mu \frac{\xi - \lambda\sqrt{\psi\varphi + \nu^2}}{\xi - \mu\sqrt{\psi\varphi + \nu^2}} \mathcal{H}(\lambda) - \lambda \frac{\xi + \mu\sqrt{\psi\varphi + \nu^2}}{\xi + \lambda\sqrt{\psi\varphi + \nu^2}} \mathcal{H}(\mu) \right).$$

# Local realization

$$\mathcal{E}(\lambda) = \frac{\lambda}{\sqrt{\psi\varphi + v^2}} \sum_{m=1}^N \frac{\xi - \alpha_m \sqrt{\psi\varphi + v^2}}{\xi - \lambda \sqrt{\psi\varphi + v^2}}$$

$$\times \frac{2(v - \sqrt{\psi\varphi + v^2})S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2v(v - \sqrt{\psi\varphi + v^2})}{\psi} S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)},$$

$$\mathcal{F}(\lambda) = \frac{-\lambda}{\sqrt{\psi\varphi + v^2}} \sum_{m=1}^N \frac{\xi + \alpha_m \sqrt{\psi\varphi + v^2}}{\xi + \lambda \sqrt{\psi\varphi + v^2}}$$

$$\times \frac{2(v + \sqrt{\psi\varphi + v^2})S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2v(v + \sqrt{\psi\varphi + v^2})}{\psi} S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)},$$

$$\mathcal{H}(\lambda) = \frac{\lambda}{\sqrt{\psi\varphi + v^2}} \sum_{m=1}^N \frac{2vS_m^3 + \psi S_m^+ + \varphi S_m^-}{(\lambda - \alpha_m)(\lambda + \alpha_m)}.$$

# “Vacuum” eigenstate

- Generating function is:

$$\tau(\lambda) = \text{tr}_0 \mathcal{L}_0^2(\lambda) = \mathcal{H}^2(\lambda) + \frac{1}{2} (\mathcal{E}(\lambda)\mathcal{F}(\lambda) + \mathcal{F}(\lambda)\mathcal{E}(\lambda))$$

- “All-spins-up” state  $\Omega_+$  is an eigen&vacuum state:

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H} \quad S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0$$

- since  $\mathcal{E}(\lambda) \Omega_+ = 0$        $\mathcal{H}(\lambda) \Omega_+ = \rho(\lambda) \Omega_+$        $\tau(\lambda) \Omega_+ = \chi_0(\lambda) \Omega_+$
- ...

**But it is not!!!**

**Unless we reduce generality and set  $\psi=0$  ???...**

# New (general) vacuum state

- Define:

$$\omega_m = \begin{pmatrix} \psi^2 \\ -\sqrt{2} \psi (v - \sqrt{\psi\varphi + v^2}) \\ (v - \sqrt{\psi\varphi + v^2})^2 \end{pmatrix} \in \mathbb{C}^3 = V_m \quad \Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}$$

- so that:  $\left( 2(v - \sqrt{\psi\varphi + v^2}) S_m^3 + \psi S_m^+ - \frac{\psi\varphi + 2v(v - \sqrt{\psi\varphi + v^2})}{\psi} S_m^- \right) \omega_m = 0,$

$$(2v S_m^3 + \psi S_m^+ + \varphi S_m^-) \omega_m = 2\sqrt{\psi\varphi + v^2} \omega_m.$$

- i.e. so that now really:

$$\mathcal{E}(\lambda) \Omega_+ = 0 \quad \text{and} \quad \mathcal{H}(\lambda) \Omega_+ = \rho(\lambda) \Omega_+ \quad \text{with} \quad \rho(\lambda) = \sum_{m=1}^N \frac{2\lambda}{\lambda^2 - \alpha_m^2}$$

$$\tau(\lambda) \Omega_+ = \chi_0(\lambda) \Omega_+ \quad \text{with} \quad \chi_0(\lambda) = \rho^2(\lambda) + \frac{\xi^2 + (\psi\varphi + v^2)\lambda^2}{\xi^2 - (\psi\varphi + v^2)\lambda^2} \frac{\rho(\lambda)}{\lambda} - \rho'(\lambda)$$



To get rid of “unwanted terms” we impose Bethe equations:

$$\rho(\mu_j) + \frac{(\psi\varphi + v^2)\mu_j}{\xi^2 - (\psi\varphi + v^2)\mu_j^2} - \sum_{k \neq j}^M \frac{2\mu_j}{\mu_j^2 - \mu_k^2} = 0$$

- We show that the eigenvalues are real! s!

- General solution can be written as:

$$\Phi_M(\mu_1, \mu_2, \dots, \mu_M) = \mathcal{F}(\mu_1)\mathcal{F}(\mu_2) \cdots \mathcal{F}(\mu_M)\Omega_+,$$

- since:

$$\tau(\lambda)\Phi_M(\mu_1, \mu_2, \dots, \mu_M) = \chi_M(\lambda, \mu_1, \mu_2, \dots, \mu_M) \Phi_M(\mu_1, \mu_2, \dots, \mu_M)$$

$$+ \sum_{j=1}^M \frac{4\lambda}{\lambda^2 - \mu_j^2} \frac{\xi - \mu_j \sqrt{\psi\varphi + v^2}}{\xi - \lambda \sqrt{\psi\varphi + v^2}} \left( \rho(\mu_j) + \frac{(\psi\varphi + v^2)\mu_j}{\xi^2 - (\psi\varphi + v^2)\mu_j^2} - \sum_{k \neq j}^M \frac{2\mu_j}{\mu_j^2 - \mu_k^2} \right) \times$$

$$\times \Phi_M(\lambda, \mu_1, \dots, \hat{\mu}_j, \dots, \mu_M),$$

- eigenvalues:

$$\chi_M(\lambda, \mu_1, \mu_2, \dots, \mu_M) = \chi_0(\lambda) - \sum_{j=1}^M \frac{4\lambda}{\lambda^2 - \mu_j^2} \left( \rho(\lambda) + \frac{(\psi\varphi + v^2)\lambda}{\xi^2 - (\psi\varphi + v^2)\lambda^2} - \sum_{k \neq j}^M \frac{\lambda}{\lambda^2 - \mu_k^2} \right).$$

# Action of Gaudin Hamiltonians

- Full off-shell action:

$$\begin{aligned}
 H_m \Phi_M(\mu_1, \mu_2, \dots, \mu_M) &= \mathcal{E}_{m,M} \Phi_M(\mu_1, \mu_2, \dots, \mu_M) + \sum_{j=1}^M \frac{4\alpha_m}{\alpha_m^2 - \mu_j^2} \frac{\xi - \mu_j \sqrt{\psi\varphi + v^2}}{\xi - \alpha_m \sqrt{\psi\varphi + v^2}} \times \\
 &\times \left( \rho(\mu_j) + \frac{(\psi\varphi + v^2)\mu_j}{\xi^2 - (\psi\varphi + v^2)\mu_j^2} - \sum_{k \neq j}^M \frac{2\mu_j}{\mu_j^2 - \mu_k^2} \right) \left( \frac{-2(v + \sqrt{\psi\varphi + v^2})S_m^3 - \psi S_m^+ + \frac{\psi\varphi + 2v(v + \sqrt{\psi\varphi + v^2})}{\psi} S_m^-}{2\sqrt{\psi\varphi + v^2}} \right) \times \\
 &\times \Phi_{M-1}(\mu_1, \dots, \hat{\mu}_j, \dots, \mu_M),
 \end{aligned}$$

- energies:

$$\begin{aligned}
 \mathcal{E}_{m,M} &= \text{Res}_{\lambda=\alpha_m} \chi_M(\lambda, \mu_1, \mu_2, \dots, \mu_M) \\
 &= \frac{2\xi^2}{(\xi^2 - (\psi\varphi + v^2)\alpha_m^2)\alpha_m} + \sum_{n \neq m}^N \frac{4\alpha_m}{\alpha_m^2 - \alpha_n^2} - \sum_{j=1}^M \frac{4\alpha_m}{\alpha_m^2 - \mu_j^2}
 \end{aligned}$$

# Knizhnik-Zamolodchikov equations

- We want to find  $\Psi(\alpha_1, \alpha_2, \dots, \alpha_N)$  such that:

$$\kappa \partial_{\alpha_m} \Psi(\alpha_1, \alpha_2, \dots, \alpha_N) = \tilde{H}_m \Psi(\alpha_1, \alpha_2, \dots, \alpha_N)$$

- Must take:  $\xi = 0$

- Seek solutions in form:

$$\Psi(\alpha_1, \alpha_2, \dots, \alpha_N) = \oint \oint \dots \oint \Upsilon(\vec{\mu}; \vec{\alpha}) \cdot \tilde{\Phi}_M(\vec{\mu}; \vec{\alpha}) d\mu_1 d\mu_2 \dots d\mu_M$$

- where:

$$\kappa \partial_{\alpha_m} Y = \tilde{\mathcal{E}}_{m,M} Y,$$

$$\kappa \partial_{\mu_j} Y = \beta_M(\mu_j) Y,$$

$$\beta_M(\mu_j) = -2 \left( \rho(\mu_j) - \frac{1}{\mu_j} - \sum_{k \neq j}^M \frac{2\mu_j}{\mu_j^2 - \mu_k^2} \right)$$

We find:

$$\Psi(\alpha_1, \alpha_2, \dots, \alpha_N) = \oint \oint \dots \oint \Upsilon(\vec{\mu}; \vec{\alpha}) \cdot \tilde{\Phi}_M(\vec{\mu}; \vec{\alpha}) d\mu_1 d\mu_2 \dots d\mu_M$$

Where:

$$\begin{aligned} \Upsilon(\vec{\mu}; \vec{\alpha}) &= \exp\left(\frac{S(\vec{\mu}; \vec{\alpha})}{\kappa}\right) \\ S(\vec{\mu}; \vec{\alpha}) &= \sum_{m=1}^N \left( \sum_{n \neq m}^N \ln(\alpha_n^2 - \alpha_m^2) - \sum_{j=1}^M 2 \ln(\mu_j^2 - \alpha_m^2) \right) \\ &\quad + \sum_{j=1}^M \left( \ln(\mu_j^2) + \sum_{k \neq j}^M \ln(\mu_j^2 - \mu_k^2) \right). \end{aligned}$$

Also, neat formulas for...

...norms (on-shell):

$$\|\tilde{\Phi}_M(\mu_1, \mu_2, \dots, \mu_M)\|^2 = \det \left( \begin{array}{cccc} \frac{\partial^2 S}{\partial \mu_1^2} & \frac{\partial^2 S}{\partial \mu_1 \partial \mu_2} & \cdots & \frac{\partial^2 S}{\partial \mu_1 \partial \mu_M} \\ \vdots & \ddots & & \vdots \\ \frac{\partial^2 S}{\partial \mu_M \partial \mu_1} & \frac{\partial^2 S}{\partial \mu_M \partial \mu_2} & \cdots & \frac{\partial^2 S}{\partial \mu_M^2} \end{array} \right) \Bigg|_{\substack{\beta_M(\mu_1)=0 \\ \vdots \\ \beta_M(\mu_M)=0}}$$

...scalar products (off-shell):

$$\langle \tilde{\Phi}_M(\mu_1, \mu_2, \dots, \mu_M), \tilde{\Phi}_M(\nu_1, \nu_2, \dots, \nu_M) \rangle = 4^M \sum_{\sigma \in \mathcal{S}_M} \det \mathcal{M}^\sigma$$

$$\mathcal{M}_{jj}^\sigma = -\frac{\mu_j \rho(\mu_j) - \nu_{\sigma(j)} \rho(\nu_{\sigma(j)})}{\mu_j^2 - \nu_{\sigma(j)}^2} - \sum_{k \neq j} \frac{\mu_k^2 + \nu_{\sigma(k)}^2}{(\mu_j^2 - \mu_k^2)(\nu_{\sigma(j)}^2 - \nu_{\sigma(k)}^2)},$$

$$\mathcal{M}_{jk}^\sigma = -\frac{\mu_k^2 + \nu_{\sigma(k)}^2}{(\mu_j^2 - \mu_k^2)(\nu_{\sigma(j)}^2 - \nu_{\sigma(k)}^2)}, \quad \text{for } j, k = 1, 2, \dots, M.$$

## To summarize...

- We solved ABA for rational  $so(3)$  Gaudin model with **fully general** nontrivial boundary conditions (thanks to nontrivial choice of vacuum)
- Found solutions for Knizhnik-Zamolodchikov equations
- Presented closed-form formulas for (on-shell) norms and (off-shell) scalar products of Bethe vectors.

Thank you.