Higher category theory and n-groups as gauge symmetries for quantum gravity

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INTRODUCTION

A short recap of the spinfoam quantization method:

- Step 1: Rewrite the GR action — as a topological BF theory plus simplicity constraint,

$$S_{\text{Plebanski}}[B,\omega,\phi] = \int_{\mathcal{M}_4} \langle B \wedge F(\omega) \rangle_{\mathfrak{g}} + \langle \phi(B \wedge B) \rangle_{\mathfrak{g}} \,,$$

  where the Lie group G is Lorentz-like, and $\mathfrak{g}$ is its Lie algebra.

- Step 2: Quantize the topological sector — a state sum over a triangulated manifold $T(\mathcal{M}_4)$,

$$Z_{BF} = \sum_{\Lambda} \quad \prod_{v} \mathcal{A}_v(\Lambda) \prod_{e} \mathcal{A}_e(\Lambda) \prod_{\Delta} \mathcal{A}_{\Delta}(\Lambda) \prod_{\tau} \mathcal{A}_{\tau}(\Lambda) \prod_{\sigma} \mathcal{A}_{\sigma}(\Lambda) \,.$$

  "Colors" Λ are reps of G, amplitudes $\mathcal{A}$ chosen so that Z_{BF} is invariant wrt. Pachner moves.

- Step 3: Impose the simplicity constraint — deform the invariant Z_{BF} by modifying the amplitudes and reps,

$$Z_{BF} \to Z_{GR} : \qquad \mathcal{A}(\Lambda) \to W(j) \,, \qquad j = f(\Lambda) \,,$$

  obtaining a state sum Z_{GR} which defines a spinfoam model (Barret-Crane, EPRL/FK, etc).

Key question — how to add matter fields into the above?

HIGHER CATEGORY THEORY

A flash introduction to higher category theory:

- An n-category is a set of *objects* with:
  - *morphisms* (maps between objects),
  - 2-*morphisms* (maps between morphisms),
  - 3-*morphisms* (maps between 2-morphisms), ... up to n-*morphisms*,

  along with certain axioms to provide suitable rules for composition, associativity, etc.
- An n-group is a special case of an n-category, which has only one object, and all morphisms are invertible.

An introduction to higher category theory (for physicists):

$\Rightarrow$ look up **"An Invitation to Higher Gauge Theory"** [Baez, Huerta (2011)]

The purpose of n-groups (for physicists):

$\Rightarrow$ *more fine-grained description of symmetry* using an n-group, than using a group,

$\Rightarrow$ generalization of differential geometry: *parallel transport, connection, holonomy, curvature.*

LIE 3-GROUPS

Focus on a strict Lie 3-group, isomorphic to a 2-crossed module:

[Faria Martins, Picken (2011); Wang (2014)]

$$\left(\quad L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \quad , \quad \triangleright \quad , \quad \{\_\,,\_\} \quad \right)$$

- L, H, G — Lie groups,
- δ, ∂ — boundary morphisms,
- $\triangleright$ — action of G, $\quad \triangleright : G \times G \to G, \quad \triangleright : G \times H \to H, \quad \triangleright : G \times L \to L,$
- $\{\_\,,\_\}$ — Peiffer lifting, $\quad \{\_\,,\_\} : H \times H \to L.$

Axioms that hold among these maps:

Chain complex:	$\partial\delta = 1_G,$
Conjugation:	$g \triangleright g_0 = g\, g_0\, g^{-1},$
G-equivariance of ∂ and δ:	$g \triangleright \partial h = \partial(g \triangleright h)\,, \quad g \triangleright \delta l = \delta(g \triangleright l),$
G-equivariance of lifting:	$g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\},$
Peiffer commutator:	$\delta\,\{h_1, h_2\} = h_1 h_2 h_1^{-1} (\partial h_1) \triangleright h_2^{-1},$
L-commutator:	$\{\delta l_1, \delta l_2\} = l_1 l_2 l_1^{-1} l_2^{-1},$
δ-lifting relation:	$\{\delta l, h\}\,\{h, \delta l\} = l(\partial h \triangleright l^{-1}),$
Left product rule:	$\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\}\,\partial h_1 \triangleright \{h_2, h_3\}.$

LIE 3-GROUPS

Purpose of all this — to generalize the notion of parallel transport, from curves to surfaces to volumes:

- Connection generalized to a 3-connection (α, β, γ), a triple of algebra-valued differential forms:

$$\begin{array}{rcllr} \alpha & = & \alpha^{\alpha}{}_{\mu}(x) & \tau_{\alpha} \otimes \mathbf{d}x^{\mu} & \in \mathfrak{g} \otimes \Lambda^1(\mathcal{M})\,, \\ \beta & = & \frac{1}{2}\,\beta^{a}{}_{\mu\nu}(x) & t_a \otimes \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu} & \in \mathfrak{h} \otimes \Lambda^2(\mathcal{M})\,, \\ \gamma & = & \frac{1}{3!}\,\gamma^{A}{}_{\mu\nu\rho}(x) & T_A \otimes \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu} \wedge \mathbf{d}x^{\rho} & \in \mathfrak{l} \otimes \Lambda^3(\mathcal{M})\,. \end{array}$$

- Line holonomy generalized to surface and volume holonomies:

$$g = \mathcal{P}\mathrm{exp} \int_{\mathcal{P}_1} \alpha\,, \qquad h = \mathcal{S}\mathrm{exp} \int_{\mathcal{S}_2} \beta\,, \qquad l = \mathcal{V}\mathrm{exp} \int_{\mathcal{V}_3} \gamma\,.$$

- Ordinary curvature generalized to 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, where:

$$\begin{array}{rcl} \mathcal{F} & = & \mathbf{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \\ \mathcal{G} & = & \mathbf{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\ \mathcal{H} & = & \mathbf{d}\gamma + \alpha \wedge^{\triangleright} \gamma - \{\beta \wedge \beta\}\,. \end{array}$$

HIGHER GAUGE THEORY

At this point one can construct the action for a higher gauge theory:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

$\Rightarrow$ Topological $3BF$ theory, based on the 3-group $(\, L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G \,,\, \triangleright \,,\, \{\_\,,\_\} \,)$.

The physical interpretation of the Lagrange multipliers C and D:

- for $H = \mathbb{R}^4$, multiplier C can be ***interpreted as the tetrad*** 1***-form***:

$$C \quad \to \quad e \quad = \quad e^a{}_\mu(x)\, t_a \otimes \mathbf{d}x^\mu \,,$$

[Miković, MV (2012)]

- for given L, multiplier D can be ***interpreted as the set of matter fields***:

$$D \quad \to \quad \phi \quad = \quad \phi^A(x)\, T_A \,.$$

[Radenković, MV (2019)]

$\Rightarrow$ The action thus becomes:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle e \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle \phi \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

THE STANDARD MODEL

How many real-valued field components do we have in the Standard Model?

The fermion sector gives us:

$$\left.\begin{matrix} \begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L & \begin{pmatrix}u_r \\ d_r\end{pmatrix}_L & \begin{pmatrix}u_g \\ d_g\end{pmatrix}_L & \begin{pmatrix}u_b \\ d_b\end{pmatrix}_L \\ \\ (\nu_e)_R & (u_r)_R & (u_g)_R & (u_b)_R \\ \\ (e^-)_R & (d_r)_R & (d_g)_R & (d_b)_R \end{matrix}\right\} = 16 \ \frac{\text{spinors}}{\text{family}} \times$$

$$\times 3 \text{ families } \times 4 \ \frac{\text{real-valued components}}{\text{spinor}} \ = 192 \text{ real-valued components } \phi^A \, .$$

The Higgs sector gives us:

$$\left.\begin{pmatrix}\phi^+ \\ \phi_0\end{pmatrix}\right\} = 2 \text{ complex scalar fields } = 4 \text{ real-valued components } \phi^A \, .$$

This suggests the structure for L in the form:

$$L = \mathbb{C}^4 \times \mathbb{G}^{64} \times \mathbb{G}^{64} \times \mathbb{G}^{64} \, ,$$

where $\mathbb{G}$ is the Grassmann algebra.

THE STANDARD MODEL

The actions $\triangleright : G \times L \to L$ and $\triangleright : G \times H \to H$ specify the transformation properties of matter ϕ^A and tetrad $e^a{}_\mu$ with respect to Lorentz and internal symmetries:

- Choose the group $G = SO(3,1) \times SU(3) \times SU(2) \times U(1)$. Then, for example, given any $g \in G$ and a doublet

$$\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L ,$$

  the action $g \triangleright u_b$ encodes that u_b consists of 4 real-valued fields which transform as:

  - a left-handed spinor wrt. $SO(3,1)$,
  - as a "blue" component of the fundamental representation of $SU(3)$,
  - and as "isospin $+\frac{1}{2}$" of the left doublet wrt. $SU(2) \times U(1)$.

- Next choose the group $H = \mathbb{R}^4$. The action $\triangleright$ of G on H is via vector representation for the $SO(3,1)$ part and via trivial representation for the $SU(3) \times SU(2) \times U(1)$ part.

- Finally, the other maps in the 3-group are chosen to be trivial. For all $l \in L$ and $\vec{u}, \vec{v} \in H$,

$$\delta l = 1_H = 0 \,, \qquad \partial \vec{v} = 1_G \,, \qquad \{\vec{u}, \vec{v}\} = 1_L \,.$$

THE STANDARD MODEL

The Standard Model **3-group,** $(\, L \xrightarrow{\delta} H \xrightarrow{\partial} G \,,\, \triangleright \,,\, \{\_\,,\_\} \,)$**, defined as:**

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1)\,, \qquad H = \mathbb{R}^4\,,$$

$$L = \mathbb{C}^4 \times \mathbb{G}^{64} \times \mathbb{G}^{64} \times \mathbb{G}^{64}\,.$$

The constrained 3BF action for the Standard Model coupled to Einstein-Cartan gravity:

$$\begin{aligned}
S \;=\;& \int \overbrace{B_\alpha \wedge F^\alpha + B^{[ab]} \wedge R_{[ab]}}^{\langle B\wedge\mathcal{F}\rangle} + \overbrace{e_a \wedge \nabla\beta^a}^{\langle C\wedge\mathcal{G}\rangle} + \overbrace{\phi^A(\nabla\gamma)_A + \bar\psi_A(\vec\nabla\gamma)^A - (\bar\gamma\,\overleftarrow{\nabla})_A\psi^A}^{\langle D\wedge\mathcal{H}\rangle} && \text{3BF} \\
&- \int \lambda_{[ab]} \wedge \left(B^{[ab]} - \frac{1}{16\pi l_p^2}\,\varepsilon^{[ab]cd}\, e_c \wedge e_d \right) + \frac{1}{96\pi l_p^2}\,\Lambda\,\varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d && \text{GR and CC} \\
&+ \int \lambda^\alpha \wedge \left(B_\alpha - 12\, C_\alpha{}^\beta M_{\beta ab}\, e^a \wedge e^b \right) + \zeta^{\alpha ab} \left(M_{\alpha ab}\,\varepsilon_{cdef}\, e^c \wedge e^d \wedge e^e \wedge e^f - F_\alpha \wedge e_a \wedge e_b \right) && \text{YM} \\
&+ \int \lambda^A \wedge \left(\gamma_A - H_{abcA}\, e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA}\,\varepsilon^{cdef}\, e_d \wedge e_e \wedge e_f - (\nabla\phi)_A \wedge e_a \wedge e_b \right) && \text{Higgs} \\
&- \int \frac{1}{12}\,\chi \left(\phi^A\phi_A - v^2 \right)^2 \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d && \text{Higgs potential} \\
&+ \int \bar\lambda_A \wedge \left(\gamma^A + \frac{i}{6}\,\varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \left(\gamma^d\psi \right)^A \right) - \lambda^A \wedge \left(\bar\gamma_A - \frac{i}{6}\,\varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \left(\bar\psi\gamma^d \right)_A \right) && \text{Dirac} \\
&- \int \frac{1}{12}\, Y_{ABC}\, \bar\psi^A \psi^B \phi^C\, \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d && \text{Yukawa} \\
&+ \int 2\pi i\, l_p^2\, \bar\psi_A \gamma_5 \gamma^a \psi^A\, \varepsilon_{abcd}\, e^b \wedge e^c \wedge \beta^d\,. && \text{spin-torsion}
\end{aligned}$$

Finally, there is also a 4-group generalization, with a $4BF$ **action.** [Miković, MV (2021)]

QUANTIZATION

Revisit the spinfoam quantization method:

- Step 1: Rewrite the GR+SM action… **done!**
- Step 2: Quantize the topological sector… **done!** (almost) [Radenković, MV (2022)]

$$
\begin{aligned}
Z \;=\;& |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\,|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|} \\
&\times \prod_{(jk)\in\Lambda_1}\int_G dg_{jk} \prod_{(jk\ell)\in\Lambda_2}\int_H dh_{jk\ell} \prod_{(jk\ell m)\in\Lambda_3}\int_L dl_{jk\ell m} \\
&\times \prod_{(jk\ell)\in\Lambda_2} \delta_G\Big(\partial(h_{jk\ell})\,g_{k\ell}\,g_{jk}\,g_{j\ell}^{-1}\Big) \prod_{(jk\ell m)\in\Lambda_3} \delta_H\Big(\delta(l_{jk\ell m})h_{j\ell m}\,(g_{\ell m}\rhd h_{jk\ell})\,h_{k\ell m}^{-1}\,h_{jkm}^{-1}\Big) \\
&\times \prod_{(jk\ell mn)\in\Lambda_4} \delta_L\Big(l_{j\ell mn}^{-1}\,h_{j\ell n}\rhd'\{h_{\ell mn},(g_{mn}g_{\ell m})\rhd h_{jk\ell}\}_{\mathrm{p}}\,l_{jk\ell n}^{-1}(h_{jkn}\rhd' l_{k\ell mn})l_{jkmn}h_{jmn}\rhd'(g_{mn}\rhd l_{jk\ell m})\Big)\,.
\end{aligned}
$$

  Invariant wrt. $4D$ Pachner moves!

- Step 3: Impose the simplicity constraints… **work in progress!**

QUANTIZATION

GR without matter can be described using $\boldsymbol{2}$***-groups*** $(H \stackrel{\partial}{\to} G, \triangleright)$***:***

- Topological $2BF$ theory developed and studied: [Girelli, Pfeiffer, Popescu (2008)]
  [Miković, Martins (2011)]

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge \mathcal{F}_{ab} + C^a \wedge \mathcal{G}_a \,.$$

- The choice $G = SO(3,1)$, $H = \mathbb{R}^4$, is called the *Poincaré* 2-*group*. The action for GR is

$$S_{GR} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab}(\omega) + e^a \wedge G_a - \phi_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd}\, e_c \wedge e_d \right) .$$

  One possible quantization prescription leads to the *spincube model.* [Miković, MV (2012)]

- Representation theory for 2-groups (including the Poincaré 2-group), has been developed in great detail. [Baez, Baratin, Freidel, Wise (2012)]

- The topological invariant and TQFT for the *Euclidean* 2-*group* ($G = SO(4)$, $H = \mathbb{R}^4$) has also been studied in detail. [Baratin, Freidel (2015)]
  [Asante, Dittrich, Girelli, Riello, Tsimiklis (2020)]

CONCLUSIONS

- Higher gauge theory represents a formalism where gravity, gauge fields, fermions and Higgs are treated on an equal footing.
- Resulting generalized spinfoam models naturally include matter fields coupled to gravity.
- The underlying algebraic structure of a 3-group classifies all fundamental fields by specifying groups L, H, G and their maps $\delta, \partial, \triangleright, \{\_\,,\_\}$.
- This structure has natural geometrical interpretation of parallel transport along a curve, a surface, and a volume.
- The gauge group L specifies the complete matter sector of the Standard Model if one chooses

$$L = \mathbb{C}^4 \times \mathbb{G}^{64} \times \mathbb{G}^{64} \times \mathbb{G}^{64}\,.$$

- The action $\triangleright$ of G on L specifies the transformation properties of matter fields.
- Nontrivial choices of the 3-group structure may provide new avenues for research on unification of all fields.

THANK YOU!